Identification of the tensile force in tie-rods using modal analysis tests.
DECLARATION

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... danh tăng bỏ, mẹ, chỉ 

và hai cháu sinh đôi thân yêu của đi
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Abstract

Metallic tie-rods are often used for masonry arches and vaults to provide the lateral restraints and contribute fundamentally to the overall structural equilibrium. In the field of restoration, structural strengthening and monitoring of those ancient structures, it is important to identify the tensile forces acting in the tie-rods. This work studies a methodology for experimentally estimation of service stress levels in tie-rods by using modal analysis tests as non-destructive tests in historical constructions.

At first, a compilation of a state of the art of ancient tie-rods is presented to provide the common characteristics of ancient tie-rods. Then, a numerical model was developed for axially loaded tie-rod. A parametric study was carried out with a total of 432 models. Based on the results of the parametric study, a methodology using theoretical equation and several standard charts were proposed to determine a range of tensile stress of tie-rods with different characteristics.

To validate the results of the numerical-and-theoretical-based methodology and charts, 84 experimental modal analysis tests were performed for four specimens in the laboratory. For comparison, three different types of sensors: conventional, wireless and laser vibrometer were used. The wireless accelerometer and laser vibrometer system provide accurate results compared with the wired accelerometer.

After that, the results of the experiments were calibrated with the numerical results to identify the tensile stress, rotational stiffness of the supports and elastic modulus of tie-rods, as well as to assess several types of experimental uncertainties. From the results of the calibration process, the most suitable analysis method which should be used to describe accurately the dynamic response of tie-rods and its tensile stress was proposed. The analysis method takes into account the bending curvature due to self-weight of tie-rod. Finally, the methodology to estimate a range of tensile stress in tie-rods using theoretical formula or proposed standard charts was concluded. In addition, two techniques to identify in-situ the tensile stress in tie-rods were discussed. The techniques are based on frequency-based identification methods that minimize the measurement error. They not only give the value of tensile stress but also the estimation error of the identification of tensile stress.
Resumo

Tópico: “Determinação da força de tracção em tirantes usando ensaios de identificação modal”

Os tirantes metálicos foram frequentemente utilizados em arcos e abóbadas de estruturas de alvenaria para garantir o travamento lateral e, sobre tudo, para assegurar o equilíbrio global da estrutura. Desta forma, no âmbito da reabilitação, reforço e monitorização de construções antigas, torna-se importante determinar a força de tracção em tirantes. Este trabalho procurou estudar metodologias para estimar experimentalmente as tensões de serviço em tirantes, usando ensaios de identificação modal como um método não destrutivo.

Primeiramente, foi realizada uma revisão do estado da arte no que se refere aos tirantes antigos para se compreender as suas características. Em seguida, foi desenvolvido um modelo numérico para tirantes sujeitos a esforços axiais, utilizado para realizar um estudo paramétrico, com um total de 432 análises. Com base no estudo paramétrico, uma metodologia usando a formulação teórica e vários ábacos foi proposta para determinar a gama de tensão de tracção em tirantes.

Para validar a metodologia, 84 ensaios modais experimentais foram realizados em quatro provetes em laboratório. Três diferentes tipos de sistemas de aquisição de dados foram estudados: com sensores convencionais com fios; com sensores sem fios; e com um sensor laser. Os sistemas com sensores sem fios e com o sensor laser obtiveram bons resultados, quando comparados com o sistema de sensores convencionais.

Depois dos ensaios, os resultados dos tirantes foram usados para calibrar um modelo numérico com vista a identificar a tensão de tracção, a rigidez à rotação dos apoios e o módulo de elasticidade do material, assim como para determinar a influência de algumas incertezas nos ensaios. A partir dos resultados do processo de calibração, foi proposto um método mais adequado para representar, com precisão, a resposta dinâmica dos tirantes e determinar a tensão de tracção. O método toma em consideração a curvatura por flexão do tirante devido ao peso próprio. Finalmente, a metodologia para estimar uma gama de tensão de tracção usando a formulação teórica e ábacos foi completada. Adicionalmente, as duas técnicas para identificar a tensão de tracção em tirantes foram apresentadas. Estas técnicas não só determinam o valor da tensão de tracção, como também indicam a margem de erro da estimativa.
Tóm Luận

Chủ đề: “Xác định lực căng trong thanh kim loại bằng cách sử dụng phương thức thử nghiệm phân tích thế dạng”

Thanh kim loại thường được sử dụng cho vòng nề và hàn chế lực căng ngang và đóng góp cơ bản cho sự cân bằng cấu trúc tổng thể. Trong lĩnh vực khối phức, cũng có cơ cấu và giám sát của những công trình có, sự xác định độ bền lực kéo trong thanh kim loại mang tầm quan trọng cao. Công trình này nghiên cứu một phương pháp ước lượng của thử nghiệm mức độ lực căng trong thanh kim loại bằng thử nghiệm phân tích thế dạng không gây phá hủy trong các công trình lịch sử.

Ban đầu, một biên soạn được trình bày để hiển thị các đặc điểm chung của thanh kim loại. Sau đó, một số mô hình đã được phát triển. Một nghiên cứu tham số được thực hiện với tổng số 432 phần tích. Cần cung cấp kết quả của nghiên cứu tham số, một phương pháp sử dụng phương trình lý thuyết và biểu đồ tiêu chuẩn được xuất để xác định lực căng trong thanh kim loại với các đặc tính khác nhau.

Để xác nhận các kết quả của phương pháp số-vá-lý thuyết và biểu đồ tiêu chuẩn, 84 thí nghiệm phân tích phương thức thử nghiệm được thực hiện trong bốn miền đất trên phẳng thử nghiệm. Để so sánh, ba loại khác nhau của các biên: thông thường, không dây và tia sóng được sử dụng. Các gia tốc không dây và hệ thống tia sóng cung cấp kết quả chính xác so với gia tốc cố định.

Sau đó, các kết quả của thí nghiệm đã được hiệu chuẩn với kết quả tính toán để xác định lực căng trong thanh kim loại, không quay và mở dọn dàn hỗ, cũng như đánh giá sự chính xác của thí nghiệm. Từ kết quả của quá trình hiệu chuẩn, phương pháp phân tích phù hợp nhất để mở tả một cách chính xác phần ứng nang đóng của thanh kim loại được đề xuất. Phương pháp phân tích dựa vào độ cong ống do trọng lượng bán thân của thanh kim loại. Cuối cùng, phương pháp đề ước lượng lực căng trong thanh kim loại bằng cách sử dụng công thức lý thuyết hoặc các biểu đồ tiêu chuẩn được kết luận. Ngoài ra, hai kỹ thuật để xác định tại chỗ lực căng trong thanh kim loại được thảo luận. Các kỹ thuật dựa trên phương pháp nhận dạng tân so dựa trên giảm thiểu sai số đo.
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Glossary

List of Symbols

Δσ  Change in applied tensile stress
ΔL  Change in length
φn  Mode shapes
φ   Eigenvectors or mode shapes
ρs  Mass density
σ   Tensile stress
σ₀  Initial applied tensile stress
b   Width of the cross-section
d   Diameter of the cross-section
e   Strain
E   Elastic modulus
fₙ  Natural frequency
fₙₓₓ   Frequency of rectangular cross-section
fₙₒ   Frequency of circular cross-section
π   Pi constant
h   Height of the cross-section
I   Moment of inertia
Iₓₓ   Moment of inertia of rectangular cross-section
Iₒ   Moment of inertia of circular cross-section
k   Rotational spring stiffness of the supports
kₓₓ   Constant parameter of rectangular cross-section
kₒ   Constant parameter of circular cross-section
ωₙ  Natural circular frequency
$L$  Length

$L_{\text{eff}}$  Effective length

$L_0$  Original length

$m$  Mass per unit length

$m_{c}$  Mass per unit length of rectangular cross-section

$m_{o}$  Mass per unit length of circular cross-section

$r$  Radius of the cross-section

$T$  Tensile force

$T_{c}$  Tensile force of rectangular cross-section

$T_{o}$  Tensile force of circular cross-section

$x_{\sigma}$  Tensile stress ratio

$x_{d}$  Diameter ratio

$x_{h}$  Height ratio

$x_{f}$  Frequency ratio

$x_{f_{n}}(x_{h})$  Frequency ratio of rectangular cross-section at value of $x_{h}'$

$x_{L}$  Length ratio

$x_{\text{mode}}$  Mode ratio

$w$  Deflection
## List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>Accelerometer</td>
</tr>
<tr>
<td>BC</td>
<td>Boundary condition</td>
</tr>
<tr>
<td>CF</td>
<td>Correction factor</td>
</tr>
<tr>
<td>COMAC</td>
<td>Co-ordinate Modal Assurance Criteria</td>
</tr>
<tr>
<td>DAQ</td>
<td>Data Acquisition</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>FEMU</td>
<td>Finite Element Modal Updating method</td>
</tr>
<tr>
<td>FRS</td>
<td>Frequency Response Spectrum</td>
</tr>
<tr>
<td>LVDT</td>
<td>Linear Variable Differential Transformer</td>
</tr>
<tr>
<td>MAC</td>
<td>Modal Assurance Criteria</td>
</tr>
<tr>
<td>ND</td>
<td>Non-destructive</td>
</tr>
<tr>
<td>OMA</td>
<td>Operational Modal Analysis</td>
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Chapter 1
Introduction
1.1 Introduction

Metallic tie-rods were often used in ancient masonry buildings. They eliminate the lateral load exercised by the vaults and arches and contribute to the overall structural equilibrium. As stated by Lagomarsino and Calderini (2005), in the case of slender masonry elements, tie-rods contribute to guaranteeing an efficient connection between the constituting parts of the structure complex. Moreover, such elements play a decisive role in the control of horizontal thrusts, which are permanent in the case of arches and vaults, or environmental such as in the case of earthquakes or wind actions. It was emphasized that the stability of old masonry buildings relies on the use of metallic tie-rods.

According to Amabili et al. (2010), the tensile force on tie-rods can surpass the yield strength of the material as a consequence of foundation settlements. Especially, the strength of ancient tie-rods is not particularly high since old-time metallurgy was not able to obtain high-strength rods. Furthermore, ancient tie-rods may be damaged due to corrosion which reduces their strength.

For these reasons and the fundamental function of tie-rods, it is important to substitute the tie-rods in dangerous cases. As a result, the knowledge of the tensile forces in tie-rods is critical in the field of preservation, structural strengthening and monitoring of ancient arched and vaulted structures.

To date, several techniques have been proposed to estimate the tensile forces in metallic tie-rods using static and/or dynamic responses of the tie-rods. However, even with the cumulative efforts presented by many researchers, the estimation of tensile forces in tie-rods using dynamic tests based on the relationship with the frequency of tie-rods has not been explicitly formulated. Therefore, the development of some formulas and practical and reliable charts that can be used to accurately and rapidly assess the tensile forces in tie-rods is of great interest. Moreover, a straightforward methodology to identify experimentally the tensile forces in tie-rods is of high importance.

1.2 Objectives and Methodology

This work aims to develop a methodology for experimentally estimation of service stress levels in tie-rods by using modal analysis tests as non-destructive (ND) tests in historical constructions.
Firstly, the compilation of a state of the art related to the usage of the ancient metallic tie-rods will be studied. Based on the information collected from this literature study, the characteristics of common tie-rods will be suggested and used in the numerical simulation and parametric study. In addition, a research of the past techniques to estimate the tensile forces in the tie-rods will be elaborated to understand the current status of the subject.

Then, a numerical model is developed for axially loaded tie-rods. Parametric study will be carried out with a large number of models of different lengths and bending stiffnesses, considering different applied tensile stresses and boundary conditions. Based on the results of the parametric study, a methodology to estimate the tensile force in tie-rods will be discussed.

The results calculated from the numerical-based methodology will be validated using experimental data. The experimental modal analysis tests are performed for specimens in laboratory with three different types of sensors: conventional and wireless and laser. The results of the experiments are calibrated with numerical results to identify the tensile stresses and the boundary conditions of the tie-rods. In addition, several standard charts can be proposed to determine a range of the tensile stress of tie-rods with different characteristics.

1.3 Organization of the thesis

The thesis is organized in seven chapters as follows:

Chapter 1 gives the introduction to the work, the motivation, the objectives and the methodology, as well as the outline of the thesis;

Chapter 2 presents a state of the art of the usage of ancient tie-rods and the proposed techniques to estimate the tensile force in metallic tie-rods. The study of ancient tie-rods is to determine the common types of the tie-rods used, their lengths, cross-sections, materials, how they were connected to the masonry walls or columns and also the values of applied tensile stress. At the end of this chapter, a review of the past techniques to estimate the tensile stress in metallic tie-rods is carried out;

Chapter 3 presents a theoretical study of the dynamic response of metallic tie-rods, in order to determine the parameters affecting their frequency and modes of vibration. The partial equations of motion are introduced first, followed by the analysis of undamped free vibration of tie-rods. The analysis is divided into two cases which exclude and include the axial force effects. In each case, a summary of the frequency formulas of tie-rods with different boundary conditions is given;
**Chapter 4** presents the numerical simulation and parametric study of metallic tie-rods. An introduction of the definition of numerical models is given. A large number of tie-rods of different characteristics are modelled using a Finite Element (FE) program to assess the effect of parameters on the frequency of tie-rods. The results are compared with linear beam theory from another program and with theoretical approach. After that, a method to estimate the tensile force in tie-rods is concluded;

**Chapter 5** presents the experimental modal analysis tests in laboratory of four tie-rod specimens of different lengths, cross-sections and boundary conditions. The experimental results are compared with the numerical results;

**Chapter 6** presents the calibration of the experimental the numerical results. In this task, the optimization technique is used to identify the tensile stresses and the boundary conditions of the tie-rods. In addition, the development of techniques to identify in-situ the service stress levels in tie-rods by using modal analysis tests is also developed;

**Chapter 7** presents the conclusions and recommendations of the thesis.

A schematic representation of the outline of the thesis is presented in Figure 1.1.
Chapter 2
The State of The Art

Abstract

In this chapter, a state of the art of the usage of ancient tie-rods and the proposed techniques to estimate the tensile force in metallic tie-rods are presented. The study of ancient tie-rods is to determine the common types of the tie-rods used, their lengths, cross-sections, materials and how they were connected to the masonry walls or columns. The data collected from the state of the art study suggests the characteristics and correlation between different parameters of metallic tie-rods to be used in the numerical parametric study in the later Chapter. At the end of this Chapter, a review of the past techniques to estimate the tensile stress in metallic tie-rods is carried out. The advantages and disadvantages of each technique are discussed.
2.1 The Function of Tie-rods in Masonry Arch and Vault Structures

It is well known that masonry arches and vaults have high compressive strength and almost no tensile strength (Huerta, 2001). When these structures are subjected to lateral movements, as a result of earthquakes excitation or differential settlements, high tensile stresses and associated cracking can occur (Roeder-Carbo and Ayala, 2001). As mentioned by Vermelffoort (2001), in the cases of unstable condition, the horizontal components of the reaction forces must be taken by a tension member. Therefore, metallic tie-rods are often used for masonry arches and vaults to provide the lateral restraints and to contribute to the overall structural equilibrium.

Giuriani and Gubana (1995) stated that the main causes of vault decay are often due to insufficient confinement of the horizontal forces transmitted by vaults to walls. Other reasons can be material deterioration and cracks in the vertical walls, caused by local soil subsidence or excessive localized loads. Among the causes, they emphasized the problem of insufficient horizontal reactions is the most important and its resolution is fundamental to solving every other static problem.

To overpass this problem, ties should be placed at the arch springing line to directly support the horizontal forces. In some cases they can be replaced by extrados ties. This solution, already used in past centuries with steel or wooden ties, has been more frequently adopted since the 14th Century. A case study carried out by Giuriani and Gubana (1995) with more details regarding the extrados tie system will be presented in the Section 2.2.2.

Bussell (2008) illustrated the effect of tie-rods on arched floors made in brick and stone as shown in Figure 2.1. These can be found in multi-storey mills and similar 19th Century industrial and commercial buildings. Usually the arches were tied by wrought iron tie-rods which restrained the tendency of the arch springing to spread under gravity loading. If the tie-rods are absent or have been mistakenly cut out for some reason, the unbalanced thrust against (usually) an outside wall can produce local bowing (due to inadequate lateral restraint).
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure 2.1 – The function of tie-rods in arches (a) Brick ‘jack-arches’ in floor, restraint from spreading by tie-rods, (b) Arch spread and distortion of wall arising from removing of tie-rods in end bay (Bussell, 2008)

2.2 Ancient Tie-Rods

2.2.1 General Characteristics of Ancient Tie-Rods

Ancient tie-rods are different in size and shape. They are made of wrought iron or cast steel. In historical buildings, tie-rods often have small cross-sections due to the scarcity of metal and the high tensile strength of the material (Vermeltfoort, 2001). The common cross-sections are circular or rectangular. However, because ancient tie-rods were hand wrought, their cross-sections were often irregular or not perfectly uniform in their length (Lagomarsino and Calderini, 2005). Lagomarsino and Calderine (2005) also stated that the mechanical characteristics of iron obtained through non-industrial processes in the past are difficult to determine through ND tests. In addition, geometrical and mechanical properties of the cross-section may be altered due to corrosion.
Lagomarsino and Calderini (2005) analyzed several representative examples of tie-rods. In their numerical analysis, they chose two tie-rods of different slenderness considering two different stress states and different clamping levels at the supports for each. These two tie-rods have the dimensional characteristics and stress states as shown in Table 2-1 and Table 2-2 below.

Table 2-1
‘Ideal’ representative types for geometrical characteristics of ancient tie-rods (Lagomarsino and Calderini, 2005)

<table>
<thead>
<tr>
<th>Representative type – two extreme cases</th>
<th>Length, l (m)</th>
<th>Cross–section, diameter, Ø (mm)</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Long” tie-rods</td>
<td>9</td>
<td>55</td>
<td>representative of those tie-rods commonly found in church naves</td>
</tr>
<tr>
<td>“Short” tie-rods</td>
<td>3</td>
<td>25</td>
<td>representative of those tie-rods commonly found in arcades or loggias</td>
</tr>
</tbody>
</table>

Table 2-2
‘Ideal’ representative types for stress states of ancient tie-rods (Lagomarsino and Calderini, 2005)

<table>
<thead>
<tr>
<th>Representative type – two extreme cases</th>
<th>Stress, σ (MPa)</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Slack” tie-rod</td>
<td>25</td>
<td>-</td>
</tr>
<tr>
<td>“Stretched” tie-rod</td>
<td>125</td>
<td>quite a high value, especially if referred to old iron</td>
</tr>
</tbody>
</table>

The strength of ancient tie-rods is not particularly high since it was not able to obtain high-strength rods in old-time metallurgy. Table 2-3 presents the yield stress of metals with respect to different time periods.

Table 2-3
Yield stress of metals with respect to age (Pipinato, 2010)

<table>
<thead>
<tr>
<th>Time period</th>
<th>Metal</th>
<th>Yield stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>Iron</td>
<td>80 - 100</td>
</tr>
<tr>
<td>-</td>
<td>Wrought iron</td>
<td>159 - 221</td>
</tr>
<tr>
<td>-</td>
<td>Cast iron</td>
<td>137</td>
</tr>
<tr>
<td>Before 1905</td>
<td>Carbon steel</td>
<td>172</td>
</tr>
<tr>
<td>1905-1932</td>
<td>Carbon steel</td>
<td>206</td>
</tr>
<tr>
<td>1933-1963</td>
<td>Carbon steel</td>
<td>227</td>
</tr>
<tr>
<td>After 1963</td>
<td>Carbon steel</td>
<td>248</td>
</tr>
</tbody>
</table>
Regarding the end connection, ancient tie-rods are metallic bars normally fixed to stonework by means of bolts or anchor plates (Lagomarsino and Calderini, 2005). Anchor plates are made of cast iron, wrought iron or steel, they can have different shapes. The tie-rod-and-plate assembly braces the masonry wall against lateral bowing. The anchor plates as shown in Figure 2.2 and Figure 2.3 might be similar to those used with arches and vaults. However, many anchor plate assemblies were embedded in the masonry and not visible.

Figure 2.2 – Cicular tie-rod plate (photo by LaChiusa, 2002)  
Figure 2.3 – Other shapes of tie-rod anchorage - 39 Grand-Rue, Geneva, Switzerland (photo by LaChiusa, 2002)

Figure 2.4 shows a way of making end connection is to forge an eye or hook on the tie-rods.

Figure 2.4 – Ancient tower ties in the Porto Cathedral: (a) Deformed anchorage of tie T1 and (b) details of broken tie T3 (Lourenço and Ângela Melo, 2004)

Although tie-rods may appear to be clamped to masonry, Lagomarsino and Calderini (2005) indicate that such a clamp is all but un-perfect, depending on many different variables such as the material which it is clamped to, its deterioration, the thickness of the masonry and its fixing system.
Some examples of the metallic tie-rods found in the masonry structures around the world are presented in the following figures.

Figure 2.6 shows the church of S. Maria Assunta in Montesanto before its collapse due to an earthquake. It represented a very significant example of damage due to a poor connection between the masonry walls. The church showed longitudinal oscillation of the central nave, resting on very slender columns and arches, with respect to the more rigid lateral perimeter walls. Evident cracks could be seen above the springing of the arches and at the base of the columns. The thrust of the last longitudinal arch has instead provoked the expulsion of part of the facade, since the chain (metallic ties) did not connect all of the nave here but stopped after the first two arches (Lagomarsino, 1998).

Other examples of tie-rods having circular or rectangular cross-sections supporting the vaults and arcades are shown in Figure 2.5, Figure 2.7, Figure 2.8 and Figure 2.9.
An example of the rectangular tie-rods that the Author had observed is at the Vladislav Hall of the Old Royal Palace at the Prague Castle, Czech Republic. During her stay in Prague, the Author visited and made a preliminary visual inspection of part of the Castle (although no detail inspection of the tie-rods was made). The Old Royal Palace was first erected in the 12th Century and was rebuilt in Gothic style in the 14th Century. In 1485, the massive Vladislav Hall was added. Ever since, the vault of the gothic Vladislav Hall has been the witness of the most important state events. The whole Castle had been damaged during a large fire in 1541 and subsequent wars after the Bohemian Revolt. Fortunately, the Vladislav Hall’s vault has been preserved with several interventions, including the addition of the metallic tie-rods after the Bohemian Revolt as shown in Figure 2.10.


2.2.2 Case Studies of Ancient Tie-Rods – Tensile Force Estimation

In a case history studied by Binda, Lualdi and Saisi (2003), ND tests were applied to investigate the damaged condition of the Villa Litta Modignani palace near Milan, Italy. The presence and the geometry of the tie-rod system supporting the vault thrust were considered as one of the primary preservation interventions of the building. Thermovison and radar methods were used to define the morphology and detect the hidden reinforcements in the vault structures.

As the Villa was built starting in 1687 until the beginning of the 18th Century, evidence of the construction techniques applied in the 18th Century could be found. Binda, Lualdi and Saisi (2003) stated that during the 17th and 18th Centuries, tie-rods were frequently positioned in such a way to not be visible, such as on the extrados side of the vault also for aesthetical reasons. An example of one of the possible configurations, extracted from the Breymann treatise (Breymann, 1881) was then provided as shown in Figure 2.11.

From the results of the tests, a presence of an inclined rod hidden by the plaster of the vaults was clearly shown (Figure 2.12). Furthermore, its position was similar to the one provided in Figure 2.11. The inspection, however, did not give accurate information as to the size of the tie-rods or the state of conservation and their stress or strength.

Figure 2.11 – Example of tie rods reinforcement from the Breymann Treatise (Breymann, 1881)

Figure 2.12 – Thermovision applied to detect a inclined rod hidden by the plaster of the vault- the Villa Litta Modignani (Binda et al. 2003)
According to Giuriani and Gubana (1995), extrados ties can be a good solution to eliminate the horizontal push forces at the vault skew-backs, taking into account the architectural point of view and the structural strengthening principle that the original behaviour is maintained and any new structural element can be eventually removed with no damage. These ties are placed between the vault and the over-standing floor. In this case, a study of the global behaviour of the vault-vertical wall system is very important.

The extrados tie system can successfully contain the horizontal forces only if high vertical actions are present in the supporting walls, as it generally happens in historical buildings with vaults above the ground floor and two or three floors above. In case of insufficient vertical strength of the supporting walls, like in low buildings or in upper floor vaulted buildings, the equilibrium condition requires vertical ties in the walls. These can be placed in drilled holes inside of the walls or on external surfaces.

A 15th-Century loggia formed by eight cross-vaults of 8 x 8 m span was studied by Giuriani and Gubana (1995); two vaults subsided with very important deformations along the keystone line and a displacement of about 20 cm occurred. The remarkable loss of verticality of the perimeter walls was a consequence of very strong vault horizontal actions, which were not sufficiently contrasted by ancient wooden extrados ties.

As a result, new extrados ties were designed to provide equilibrium (Figure 2.13). They were tensioned before removing the vault props. The extrados ties in this study were not sufficient by themselves for the equilibrium, so that vertical reinforcement was also arranged in the perimeter walls. The global horizontal push action of each 8 m side of the vault was estimated to be 455 kN under service loads. The global vertical reinforcement action of an 8 m length side wall was calculated as 490 kN. The calibrated tensioning of the ties was obtained with a mechanical apparatus.

Figure 2.13 – Intervention on a 15th Century vault – new extrados tie system (Giuriani and Gubana, 1995)
Extrados ties were also found in the roof vaults covering the main corridors in the entrance at number 33, Via Zamboni. The vaults are divided into regular bays separated by decorative arches. The geometry of the vaults varies in the different bays from barrel vault to ribbed vaults. The majority of the arches were built of masonry and had rectangular tie-bars inserted at the extrados (Figure 2.14 and Figure 2.15).

The vertical translation of the springers causes the arches to buckle and the extrados tie-bars to bend as a consequence of the opposition that the arches offer to their free movements. The effect is significant as illustrated in Figure 2.14.

Work execution entailed that the old tie-bars were cut and replaced by new tie bars located on the plane of their minimum stiffness. Once cut, the old tie-bars gave back the deformation impression (Figure 2.16), and revealed a significant translation of the cut edges. This provided a clear indication of the relevant cutting-action constantly produced by the tie-bars on the masonry arches (Raffagli, 2005).

Figure 2.14 – Entrance Hall (Via Zamboni 33): Effect of vertical translation of the springers of the arches – deformed tie-bar (Raffagli, 2005)

Figure 2.15 – Entrance Hall (Via Zamboni 33): 1st-floor vaults with with the rectangular tie-bars inserted in the extrados of the masonry arches (Raffagli, 2005)

Figure 2.16 – Entrance Hall (Via Zamboni 33): Rectangular extrados tie-bars after the cut (Raffagli, 2005)
An example showing the intrados ties connected to the walls or piers supporting the vaults is given in a paper by Tullini and Laudiero (2008). They presented a case study of the 16th Century Palace Ludovico il Moro in Ferrara, Italy. In the south, two-story, wing of the building, there is a pavilion vault arcade, of 5.50 x 12.00 m² resting on two columns and two semi-columns anchored by four forged iron tie-rods having 30 mm diameter (Figure 2.17).

One of tie-rods (named no. 1 in Figure 2.18) exhibited a complete fracture in the proximity of a semi-column giving rise to a 28 mm drift of the abutments (accompanied by quite evident cracks at the vault intrados). The remaining rods were subjected to vibration tests to verify their possible overloading and to assess the tensile force the new rod was to be given. The tie-rods no. 2, 3, 4 showed tensile forces of 86, 90, 81kN, respectively. Hence, the new tie-rod was given a tensile force of 90kN by means of a torque wrench acting at the two ends alternatively.
2.2.3 Summary of the Characteristics of Ancient Tie-Rods

The surveys in some existing masonry structures provide the detailed characteristics of ancient tie-rods which are shown in the table below. These results will be used as reference data in the numerical parametric study in Chapter 4.

Table 2-4
Summary of the characteristics of ancient tie-rods in existing structures (*to be continued next page*)

<table>
<thead>
<tr>
<th>No.</th>
<th>Reference structure name</th>
<th>Material</th>
<th>Length, L (m)</th>
<th>Size – diameter, d or height, h &amp; width, b (mm)</th>
<th>Type of cross-section</th>
<th>d/L or h/L (m/m)</th>
<th>Boundary condition</th>
<th>Estimated tensile force (kN)/Tensile stress (MPa)</th>
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<td>01</td>
<td>San Vito - 1 Steel</td>
<td>3.11</td>
<td>13.5</td>
<td>Circular</td>
<td>0.00434</td>
<td>In masonry</td>
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<td></td>
<td>San Vito - 2</td>
<td>3.1</td>
<td>13.5</td>
<td></td>
<td>0.00435</td>
<td>masonry</td>
<td>37</td>
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<td></td>
<td>San Vito - 5</td>
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<td></td>
<td>San Pasquale - 1 Iron</td>
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### Table 2-5
Summary of the characteristics of ancient tie-rods in existing structures *(continued)*

<table>
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<th>No.</th>
<th>Reference name</th>
<th>Material</th>
<th>Length, ( L ) (m)</th>
<th>Size – diameter, ( d ) or height, ( h ) &amp; width, ( b ) (mm)</th>
<th>Type of cross-section</th>
<th>( d/L ) or ( h/L ) (m/m)</th>
<th>Boundary condition</th>
<th>Estimated tensile force (kN)/Tensile stress (MPa)</th>
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</tbody>
</table>

*Reference: 01 – (Lagomarsino and Calderini, 2005);
02 – (Tullini and Laudiero, 2008);
03 – (Amabili et al., 2010);
Remark: 03 – The length of the tie-rods inside the masonry is assumed to be 0.2 m for all rods E = 193 x 10^9 N/m², G = 74.8 x 10^9 N/m², \( \rho = 7870 \) kg/m³, \( \nu = 0.29 \).*
2.3 Proposed techniques to estimate the tensile force of the tie-rods to date

To date, several techniques have been proposed for both direct and indirect measurement of the tensile force in metallic tie-rods inserted into the masonry arches and vaults of historic buildings. Static and dynamic methods based on the experimental measurement of the vertical displacements caused by concentrated loads, accelerations, and the fundamental frequencies and periods of free vibrations have been applied to arrive at a more precise evaluation of the axial tensile forces acting on the tie-rod under examination. Each of these methods, with its corresponding experimental procedure, presents different advantages and disadvantages with regard to operating procedures, equipment requirements, and accuracy range in the determination of tie-rod’s stress.

2.3.1 Bati and Tonietti (2001)

Bati and Tonietti (2001) introduced a method and experimental procedure based on a single static test. The reference structural system consists of a bending moment-resistant tie beam with unknown restraint conditions at its ends. Therefore, the method does not require any assumptions when modeling the rod's extremities, nor does it discount the increase in pull induced by the transverse load as irrelevant. It requires the measurement of three vertical displacements under a concentrated load and the strain variations measured in three sections of the rod. The reliability of the method was verified through laboratory tests using tie-rods set within a sufficiently stiff metal framework in which the tension is known through continuous monitoring by means of a load cell. The correlation of the measurements with the actual tensile strength was checked; laboratory tests showed good agreement between the analytical estimates and experimental measurements.

2.3.2 Lagomarsino and Calderini (2005)

According to Lagomarsino and Calderini (2005), the advantages of dynamic tests over static tests for metallic tie-rods are non-destructiveness and rapid testing procedures. Moreover, the response of the structure can be well described by the frequencies and modes of vibration, which in turn depends on the stiffness, masses and boundary conditions of the structure itself. The tie-rods are considered as easily excitable free structures, so that the modal frequencies can be identified without much difficulty.
Lagomarsino and Calderini presented an algorithm to identify the axial tensile force in ancient tie-rods by using the first three natural frequencies, measured by means of a dynamic test. The tie-rod was modeled as an Euler beam of uniform cross-section, neglecting the shear deformation and rotary inertia, and was assumed to be simply supported at the ends with additional rotational springs. Besides the axial tensile force, the unknown variables are the bending stiffness of the section and the stiffness of the rotational springs. Because the characteristic equation of this structural system does not allow analytical solutions, the algorithm is an approximate numerical solution, based on a minimization procedure of a suitable error function. The robustness of the method is tested by identifying a number of ideal tie-rods, modeled by means of a FEM code. Moreover, since the homogeneity of the tie-rod's mechanical stiffness throughout its axis and the equality of the constraint conditions of its extremes are hypothesized, Lagomarsino and Calderini also verified how such hypotheses may influence the tensile force estimation. Finally, the method was tested on real tie-rods.

2.3.3 Park et al. (2006)

The dynamical identification of cable tension force has been proposed by Park et al. (2006) using a sensitivity-based methodology. It determines the tension force, flexural rigidity and axial rigidity of the cable from measured natural frequencies. The feasibility of the methodology was demonstrated using simulated data from a pin-pined beam with rotational springs. Then the method was validated using experimental data. A laboratory dynamic test was conducted on a high-tension bar. This technique is not immediately applicable to tie-rods since they cannot be modeled as cables and present uncertain constraints due to the portion of the rod inserted into the masonry wall or column.

2.3.4 Amabili et al. (2010)

Recently, Amabili et al. (2010) developed an ND technique to identify the in-situ tensile force in tie-rods. The technique uses a frequency-based identification method that minimizes the measurement error. In particular, the first natural frequencies of the tie-rods are experimentally identified by measuring the frequency response functions (FRFs) with instrumented hammer excitation; four to six natural frequencies can be easily identified with a simple test. Then, a numerical model, based on the Rayleigh–Ritz method, is developed for the axially loaded tie-rod by using the Timoshenko beam theory retaining shear deformation and rotary inertia. Amabili et al. assumed the non-uniform section of the rod and also the part of the tie-rod inserted inside the masonry wall as a simple support at the extremities inside the
walls. The constraints given to the part of the tie-rod inserted inside the masonry structure are assumed to be elastic foundations. The tensile force and the stiffness of the foundation are the unknowns. In some cases, the length of the rod inside the masonry wall can be also assumed as unknown. The numerical model is used to calculate the natural frequencies for a given set of unknowns. Then, a weighted difference between the calculated and identified natural frequencies is calculated and this difference is minimized in order to identify the unknowns, and in particular the tensile force. An estimation of the error in the identification of the force is given. The technique has been tested on five tie-rods at the ground floor of the famous castle of Fontanellato, Italy, providing reasonably accurate results. The method presented by Amabili et al. has advantages of simple execution, suitable for in-situ measurement on monumental buildings since the experimental apparatus is compact and no fixed and accurate reference is necessary. In addition, the use of redundant data with respect to the unknowns in the identification process minimize the measurement and modeling errors. Moreover, it estimates the accuracy in the identification of the axial force acting on the tie-rod. The disadvantage of the technique is the time-consuming minimization process which requires the calculations of a large set of eigenvalue problems.

2.3.5 Summary

Overall, even with the cumulative efforts presented by many researchers, the estimation of tensile forces in tie-rods based on the relationship with the frequency of tie-rods has not been explicitly formulated, especially when the effect of bending curvature due to self-weight is taken into account. The development of some formulas and practical and reliable charts that can be used to rapidly assess a range of the tensile forces in tie-rods is of great interest. Moreover, a straightforward and simple technique to accurately identify in-situ the tensile forces in tie-rods is of high importance.

In addition, the specific relationships between different variables affecting the frequency of tie-rods have not been assessed in details. The variables include the bending stiffness (cross-section), length, boundary condition and tensile stress. The relationships are such that when these variables are varied, it is uncertain how accurately the frequency will be affected.

Furthermore, the application of wireless technology and non-contact sensors for dynamically testing and measuring the frequencies of tie-rods is of great interest, compared with the conventional wired system.
Chapter 3
Dynamic response of metallic tie-rod

Abstract

In this chapter, the formulation of the partial differential equations of motion of metallic tie-rods is discussed, followed by the analysis of their undamped free vibration. There are two analysis cases: the elementary case excluding the axial-force effects, and the other case considering the axial-force effects. In each case, a summary of the frequency formulas of tie-rods with three different boundary conditions, pinned-pinned, fixed-pinned and fixed-fixed is given. Finally, a study of the past techniques that have been proposed to estimate the tensile force of the tie-rods is presented.
3.1 Introduction

This Chapter aims to study the theoretical formulation of the dynamic response of tie-rods. The analysis to obtain the theoretical formulas for the frequencies and mode shapes of tie-rods is investigated in depth. The first step of the analysis is the establishment of the partial differential equations of motion, and then it is followed by the study of undamped free vibration of tie-rods.

3.2 Partial Differential Equations of Motion

3.2.1 Flexural Deformation excluding Axial-Force Effects

The first case to be considered in the formulation of partial differential equations of motion is the straight, nonuniform tie-rod, assumed as a beam, shown in Figure 3.1a. The physical properties of this tie-rod are the flexural stiffness $EI(x)$ and the mass per unit length $m(x)$, both of which may vary arbitrarily with position $x$ along the span $L$. The transverse loading $p(x,t)$ is assumed to vary arbitrarily with position and time, and the transverse-displacement response $v(x,t)$ also is a function of these variables. The end-support conditions for the tie-rod are arbitrary, although they are pictured as simple supports for illustrative purposes.

The equation of motion of this system can readily be derived by considering the equilibrium of forces acting on the differential segment of tie-rod shown in Figure 3.1b.

![Figure 3.1 – Tie-rod subjected to dynamic loading: (a) tie-rod properties and coordinates; (b) resultant forces acting on differential element (Clough and Penzien, 1995)]
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Summing all forces acting vertically leads to the first dynamic-equilibrium relationship

\[ V(x, t) + p(x, t) \, dx - [ V(x, t) + \frac{\partial V(x, t)}{\partial x} \, dx ] - g_1(x, t) \, dx = 0 \]  \hspace{1cm} (3.1)

in which \( V(x, t) \) is the vertical force acting on the cut section and \( g_1(x, t) \, dx \) is the resultant transverse inertial force:

\[ g_1(x, t) \, dx = m(x) \, dx \frac{\partial^2 v(x, t)}{\partial t^2} \]  \hspace{1cm} (3.2)

Substituting Eq. (3.2) into (3.1) and dividing the resulting equation by \( dx \) yield

\[ \frac{\partial V(x, t)}{\partial x} = p(x, t) - m(x) \, dx \frac{\partial^2 v(x, t)}{\partial t^2} \]  \hspace{1cm} (3.3)

The loading in Eq. (3.3) is the resultant of the applied and inertial-force loadings.

The second equilibrium relationship is obtained by summing moments about point A on the elastic axis. After dropping the two second-order moment terms involving the inertia and applied loadings, one gets

\[ M(x, t) + V(x, t) \, dx - [ M(x, t) + \frac{\partial M(x, t)}{\partial x} \, dx ] = 0 \]  \hspace{1cm} (3.4)

Because rotational inertia is neglected, this equation simplifies directly to the standard static relationship between shear and moment

\[ \frac{\partial M(x, t)}{\partial x} = V(x, t) \]  \hspace{1cm} (3.5)

Differentiating this equation with respect to \( x \) and substituting the result into Eq. (3.3) give

\[ \frac{\partial^2 M(x, t)}{\partial x^2} + m(x) \, dx \frac{\partial^2 v(x, t)}{\partial t^2} = p(x, t) \]  \hspace{1cm} (3.6)

which, upon introducing the basic moment-curvature relationship, becomes

\[ \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 v(x, t)}{\partial x^2} \right] + m(x) \, dx \frac{\partial^2 v(x, t)}{\partial t^2} = p(x, t) \]  \hspace{1cm} (3.7)

This is the partial differential equation of motion for the elementary case of tie-rod flexure. The solution of this equation must satisfy the prescribed boundary conditions at \( x = 0 \) and \( x = L \).
3.2.2 Flexural Deformation including Axial-Force Effects

If the tie-rod considered in the above case is subjected to a time-invariant axial loading in the horizontal direction as shown in Figure 3.2a in addition to the lateral loading shown in Figure 3.1, the local equilibrium of forces is altered. The summing of the vertical forces is not affected; hence Eq. (3.3) is still valid. However, the moment-equilibrium equation now becomes

\[ M(x,t) + V(x,t) \, dx + T(x) \frac{\partial v(x,t)}{\partial x} \, dx - \left[ M(x,t) + \frac{\partial M(x,t)}{\partial x} \right] \, dx = 0 \]  

(3.8)

from which the vertical section force \( V(x,t) \) is found to be

\[ V(x,t) = - T(x) \frac{\partial v(x,t)}{\partial x} + \frac{\partial M(x,t)}{\partial x} \]  

(3.9)

Figure 3.2 – Tie-rod with static axial loading and dynamic lateral loading: (a) tie-rod deflected due to loadings; (b) resultant forces acting on differential element (Clough and Penzien, 1995)
Substituting Eq. (3.9) into Eq. (3.3) and proceeding as before, one obtains the following partial differential equation of motion, including axial-force effects:

\[ \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 v(x,t)}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ T(x) \frac{\partial v(x,t)}{\partial x} \right] + m(x) dx \frac{\partial^2 v(x,t)}{\partial t^2} = p(x,t) \]  

(3.10)

Comparing Eqs. (3.7) and (3.10), it is evident that the longitudinal loading producing the internal axial-force distribution gives rise to an additional effective transverse loading acting on the tie-rod.

3.3 Analysis of Undamped Free Vibration of Tie-rods

3.3.1 Flexural Deformation excluding Axial-Force Effects

In this Section, the undamped mode shapes and frequencies are evaluated. Because of the mathematical complications of treating systems having variable properties, the following discussion will be limited to tie-rods having uniform properties along their lengths.

First, consider the case without axial force presented in section 3.1.1 with \( EI(x) \) and \( m(x) \) set equal to constants \( EI \) and \( \bar{m} \), respectively. As shown by Eq. (3.7), the free-vibration equation of motion of this system is

\[ EI \frac{\partial^4 v(x,t)}{\partial x^4} + \bar{m} \frac{\partial^2 v(x,t)}{\partial t^2} = 0 \]  

(3.11)

Dividing by \( EI \) gives

\[ \frac{v^{\phi}}{E I} \phi(x,t) + \bar{m} \dot{\phi}(x,t) = 0 \]  

(3.12)

By separating of variables for displacement, one obtains

\[ v(x,t) = \phi(x) Y(t) \]  

(3.13)

in which \( \phi(x) \) is a specific mode shape of the free-vibration motion and \( Y(t) \) is its time-dependent amplitude. Substituting this equation in to Eq. (3.12) and dividing by \( \phi(x) Y(t) \) gives

\[ \frac{\phi^{\dot{v}}}{\phi(x)} + \frac{\bar{m}}{E I} \frac{\ddot{Y}(t)}{Y(t)} = 0 \]  

(3.14)
The solution for this equation can only be obtained if each term is a constant in accordance with

\[
\frac{\phi^iv(x)}{\phi(x)} = -\frac{m}{E.I} \ddot{Y}(t) = \alpha^4
\]  

(3.15)

Eq. (3.15) yields two ordinary differential equations

\[
\ddot{Y}(t) + \omega^2 Y(t) = 0 \quad \text{(3.16a)}
\]

\[
\phi^iv(x) - \alpha^4 \phi(x) = 0 \quad \text{(3.16b)}
\]

in which the frequency is defined as

\[
\omega^2 = \frac{\alpha^4 E I}{m} \quad \text{(i.e. } \alpha^4 = \frac{\omega^2 m}{E I}) \]  

(3.17)

Solving Eq. (3.16b) following the procedures described in (Clough and Penzien 1995), the mode shape is

\[
\phi(x) = A_1 \cos \alpha x + A_2 \sin \alpha x + A_3 \cosh \alpha x + A_4 \sinh \alpha x \]  

(3.18)

where \(A_1, A_2, A_3\) and \(A_4\) are constants which must be evaluated so as to satisfy the known boundary conditions (displacement, slope, moment, or shear) at the ends of the beam.

3.3.1.1 Pinned-Pinned End Condition

Considering the simply-supported tie-rod with uniform section, its four boundary conditions are

\[
\phi(0) = 0 \quad M(0) = E I \phi''(0) = 0 \quad \text{(a)}
\]

\[
\phi(L) = 0 \quad M(L) = E I \phi''(L) = 0 \quad \text{(b)}
\]

therefore

\[
\phi(0) = A_1 \cos 0 + A_2 \sin 0 + A_3 \cosh 0 + A_4 \sinh 0 = 0 \quad \text{(c)}
\]

\[
\phi''(0) = \alpha^2 (-A_1 \cos 0 - A_2 \sin 0 + A_3 \cosh 0 + A_4 \sinh 0) = 0
\]

Making use of \(\cos 0 = \cosh 0 = 1\) and \(\sin 0 = \sinh 0 = 0\), from the two equations above, \(A_1 = A_3 = 0\). Similarly

\[
\phi(L) = A_2 \sin \alpha L + A_4 \sinh \alpha L = 0 \quad \text{(d)}
\]

\[
\phi''(L) = \alpha^2 (-A_2 \sin \alpha L + A_4 \sinh \alpha L) = 0
\]
After setting $A_1$ and $A_3$ equal to zero, adding these two equations and cancelling out $\alpha^2$, gives $2A_4 \sinh \alpha L = 0 \Rightarrow A_4 = 0$. Only $A_2$ remains as a nonzero constant, therefore

$$\phi(x) = A_2 \sin \alpha x$$

which is the system frequency equation; it requires that

$$\alpha = n \pi / L \quad n = 0, 1, 2,\ldots\quad (g_1)$$

Substituting this expression into Eq. (3.17) and taking the square root of both sides yield the frequency expression

$$\omega_n = n \pi^2 \sqrt{\frac{EI}{mL^4}} \quad n = 1, 2,\ldots\quad (h_1)$$

The corresponding vibration mode shapes are now given by Eq. (e_1) upon substitution of Eq. (g_1) for the frequency parameter $\alpha$ in the sine term; thus, ignoring the trivial case $n = 0$, one obtains

$$\phi_n(x) = \sin \frac{n \pi x}{L} \quad n = 1, 2,\ldots\quad (i_1)$$

### 3.3.1.2 Fixed-Pinned End Condition

Considering the tie-rod with uniform section fixed at one end and pinned at the other end, its four boundary conditions are

$$\phi(0) = 0 \quad EI \phi'(0) = 0 \quad (a_2)$$

$$\phi(L) = 0 \quad M(L) = EI \phi''(L) = 0 \quad (b_2)$$

therefore

$$\phi(0) = A_1 \cos 0 + A_2 \sin 0 + A_3 \cosh 0 + A_4 \sinh 0 = 0$$

$$\phi'(0) = \alpha (-A_1 \sin 0 + A_2 \cos 0 + A_3 \sinh 0 + A_4 \cosh 0) = 0$$

$$\phi(L) = A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L = 0$$

$$\phi''(L) = \alpha^2 (-A_1 \cos \alpha L - A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L) = 0$$

$$\phi'''(L) = \alpha^3 (-A_1 \sin \alpha L + A_2 \cos \alpha L - A_3 \sinh \alpha L - A_4 \cosh \alpha L) = 0$$

$$\phi''''(L) = \alpha^4 (-A_1 \cosh \alpha L + A_2 \sinh \alpha L - A_3 \cosh \alpha L + A_4 \sinh \alpha L) = 0$$
The first two of these equations yield \( A_3 = -A_1 \) and \( A_4 = -A_2 \). Adding the last two equations, after cancelling out \( \alpha^2 \), gives \( A_3 = -A_4 \tanh \alpha L \). Substituting all these equalities into the third equation, one obtains

\[
A_1 \cos \alpha L \sinh \alpha L - A_1 \sin \alpha L \cosh \alpha L = 0
\]  

(d2)

Realizing that \( \cos \alpha L = 0 \) is not a solution for Eq. (d2) and \( \cosh \alpha L \) always > 0, for coefficient \( A_1 \) to be nonzero

\[
\tan \alpha L = \tanh \alpha L
\]  

(e2)

The solution of this equation provides the values of \( \alpha L \) which represent the frequencies of vibration of the tie-rod with fixed-pinned end condition. Plotting functions \( \tan \alpha L \) and \( \tanh \alpha L \); their crossing points give the values of \( \alpha L \) which satisfy Eq. (e2).

\[
(\alpha L)_n = \frac{\pi}{4}(2n_1 - 1) \quad n_1 = 3, 5, 7, ...
\]  

(f2)

Introducing the values of \( \alpha L \) given by Eqs. (f2) into Eq. (3.17) the corresponding circular frequencies can be obtained as shown by

\[
\omega_n = (\alpha L)_n^2 \sqrt{\frac{EI}{mL^4}} \quad n = 1, 2, ...
\]  

(g2)

The mode-shape expression of Eq. (3.18) can be written in the form

\[
\phi(x) = \cos \alpha x - \frac{\sin \alpha x \cosh \alpha x}{\sinh \alpha x}
\]  

(h2)

Substituting separately the frequency-equation roots for \( \alpha L \) into this expression, one obtains the corresponding mode-shape functions.

3.3.1.3 Fixed-Fixed End Condition

Considering the tie-rod with uniform section fixed at two ends, its four boundary conditions are

\[
\phi(0) = 0 \quad E I \phi'(0) = 0 \quad \text{(a3)}
\]

\[
\phi(L) = 0 \quad E I \phi'(L) = 0 \quad \text{(b3)}
\]

therefore
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

\[ \phi(0) = A_1 \cos 0 + A_2 \sin 0 + A_3 \cosh 0 + A_4 \sinh 0 = 0 \]
\[ \phi'(0) = \alpha (-A_1 \sin 0 + A_2 \cos 0 + A_3 \sinh 0 + A_4 \cosh 0) = 0 \]
\[ \phi(L) = A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L = 0 \]
\[ \phi'(L) = \alpha (-A_1 \sin \alpha L + A_2 \cos \alpha L + A_3 \sin \alpha L + A_4 \cosh \alpha L) = 0 \]

The first two of these equations yield \( A_3 = -A_1 \) and \( A_4 = -A_2 \). Substituting these equalities into the last two equations, and placing the resulting expressions in matrix form, one obtains

\[
\begin{bmatrix}
(c \alpha L - \cosh \alpha L) & (\sin \alpha L - \sinh \alpha L) \\
(-\sin \alpha L - \sinh \alpha L) & (c \alpha L - \cosh \alpha L)
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (d_3)

For coefficients \( A_1 \) and \( A_2 \) to be nonzero, the determinant of the square matrix in this equation must equal zero, thus giving the frequency equation

\[ 1 - \cos \alpha L \cosh \alpha L = 0 \] (e_3)

The solution of this equation \( \cos \alpha L = (1/ \cosh \alpha L) \) \( (\cosh \alpha L \neq 0) \) provides the values of \( \alpha L \) which represent the frequencies of vibration of the tie-rod with fixed-fixed end condition. The values of \( \alpha L \) which satisfy Eq. (e_3)

\[ (\alpha L)_n = \frac{\pi}{2}(2n + 1) \quad n = 1, 2, 3,... \] (f_3)

Introducing the values of \( \alpha L \) given by Eqs. (f_3) into Eq. (3.17) the corresponding circular frequencies can be obtained as shown by

\[ \omega_n = (\alpha L)_n^2 \sqrt{\frac{E I}{m L^4}} \] (g_3)

Either of Eqs. (d_3) can now be employed to express coefficient \( A_2 \) in terms of \( A_1 \); the first gives

\[ A_2 = -\frac{(\cos \alpha L - \cosh \alpha L)}{(\sin \alpha L - \sinh \alpha L)} A_1 \] (h_3)

The mode-shape expression of Eq. (3.18) can be written in the form

\[ \phi(x) = \cos \alpha x - \cosh \alpha x - \frac{(\cos \alpha x - \cosh \alpha x)}{(\sin \alpha x - \sinh \alpha x)}(\sin \alpha x - \sinh \alpha x) \] (i_3)

Substituting separately the frequency-equation roots for \( \alpha L \) into this expression, one obtains the corresponding mode-shape functions.
### 3.3.1.4 Summary of Frequency Formula of Tie-Rods Excluding Axial-Force Effects with Three Types of Boundary Conditions

Table 3-1, Table 3-2 and Table 3-3 summarize the common frequency formula and the first five mode shapes of tie-rods excluding axial-force effects with three types of boundary conditions.

#### Table 3-1
Frequency formula of tie-rods excluding axial-force effects

<table>
<thead>
<tr>
<th>System</th>
<th>Frequency formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-pinned</td>
<td>$\omega_n = \lambda_n^2 \sqrt{\frac{EI}{mL^4}}$</td>
</tr>
<tr>
<td>Fixed-pinned</td>
<td>$\left(\frac{2n_1 - 1}{4}\right)^2 \pi^2$</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td>$\left(\frac{2n + 1}{2}\right)^2 \pi^2$</td>
</tr>
</tbody>
</table>

$n = 1, 2, 3, \ldots, n_1 = 3, 5, 7, \ldots \leftrightarrow n = 1, 2, 3, \ldots$

#### Table 3-2
Frequency ratios of the first five modes of tie-rods excluding axial-force effects

<table>
<thead>
<tr>
<th>System</th>
<th>Mode 1st to 5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-pinned</td>
<td>1.0</td>
</tr>
<tr>
<td>Fixed-pinned</td>
<td>1.56 1.27 1.17 1.13 1.10</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td>2.25 1.56 1.36 1.27 1.21</td>
</tr>
</tbody>
</table>

#### Table 3-3
First five modes of tie-rods excluding axial-force effects (Pilkey, 2005)

<table>
<thead>
<tr>
<th>System</th>
<th>Mode shape for the first five modes are sketched; Nodes are located as proportion of length L measured from left</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-pinned</td>
<td></td>
</tr>
<tr>
<td>Fixed-pinned</td>
<td></td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td></td>
</tr>
</tbody>
</table>
From Table 3-2, it can be seen that the effect of boundary conditions to the frequency values is highly significant, especially for lower modes. In the case of the first mode, the frequency value is 125% higher for the fixed-fixed system than the pinned-pinned system.

The values in Table 3-2 could be used as multiplying factors to find the frequency values of other boundary conditions in case the frequency of one boundary condition is known. And for tie-rods of the same characteristics, not being subjected to axial forces, the frequency values are always in the range from 1.0 to 2.25 times the frequency values of the pinned-pinned end condition, i.e. the range of the pinned-pinned system to the fixed-fixed system.

Moreover, to examine the effect of boundary condition in terms of the effective length, $L_{\text{eff}}$, assuming that

- Pinned-pinned end condition: $L_{\text{eff}} = 1.0L$
- Fixed-pinned end condition: $L_{\text{eff}} = 0.7L$
- Fixed-fixed end condition: $L_{\text{eff}} = 0.5L$

Substituting this $L_{\text{eff}}$ into the expression for the frequency value [Eq. (3.19)], one obtains the constant multiplying factors of $1/0.7^2 = 2.04$ for fixed-pinned system and $1/0.5^2 = 4.0$ for fixed-fixed system, regardless the order of the mode.

Compare these two values with the values in Table 3-2, it can be concluded that the effect of boundary condition cannot be simply expressed in terms of the length effect. And if the length effect is used to represent the boundary condition effect, as a simplifying assumption, the difference errors should be taken into account by some corresponding correction factors, which can be calculated separately for each mode.
3.3.2 Flexural Deformation including Axial-Force Effects

Axial forces acting in a flexural element may have significant influence on the vibration behavior of the member, resulting generally in modifications of both frequencies and mode shapes. When considering free vibrations of a prismatic member having uniform physical properties, the equation of motion, including the effect of a time-invariant uniform axial force, $T$ throughout its length, is [from Eq. (3.10)]

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} - T \frac{\partial^2 v(x,t)}{\partial x^2} + \bar{m} \frac{\partial^2 v(x,t)}{\partial t^2} = 0$$

(3.20)

Separating variables leads to

$$\frac{\phi''(x)}{\phi(x)} - \frac{T}{EI} \frac{\phi'(x)}{\phi(x)} = - \frac{\bar{m}}{EI} \frac{\ddot{Y}(t)}{Y(t)} = \alpha^4$$

(3.21)

from which two independent ordinary differential equations are obtained as given by

$$\ddot{Y}(t) + \omega^2 Y(t) = 0$$

(3.22a)

$$\phi''(x) - g^2 \phi'(x) - \alpha^4 \phi(x) = 0$$

(3.22b)

in which $\omega^2$ is again defined by Eq. (3.17) and $g^2$ is given by

$$g^2 \equiv \frac{T}{EI}$$

(3.23)

The time-dependent equation [Eq. (3.22a)] show that a uniformly distributed axial force does not affect the simple harmonic character of the free vibration; however, it does affect the mode shapes and frequencies due to the presence of the term $-g^2 \phi'(x)$ in Eq. (3.22b). The solution of Eq. (3.22b) has the general form:

$$\phi(x) = Ge^{sx}$$

(3.24)

Introducing Eq. (3.24) into Eq. (3.22b) and dividing by $A e^{sx}$, it can be concluded that the exponent $s$ must satisfy the algebraic equation:

$$s^4 - g^2 s^2 - \alpha^4 = 0$$

(3.25)

(i.e. $EI s^4 - T s^2 - \omega^2 \bar{m} = 0$)

whose roots are (Lagomarsino and Calderini, 2005)
so that the general solution of Eq. (3.22b) is expressed in the form

\[
\phi(x) = C_1 e^{s_1 x} + C_2 e^{s_2 x} + C_3 e^{s_3 x} + C_4 e^{s_4 x}
\]  

(3.27)

where \( C_1, C_2, C_3 \) and \( C_4 \) are coefficients to be determined from the boundary conditions. Since the pairs of roots \( s_1, s_2 \) and \( s_3, s_4 \) are opposite each other and are, respectively, real and purely imaginary

\[
s_{1,2} = \pm \varepsilon \quad s_{3,4} = \pm \imath \delta
\]  

(3.28)

where \( \imath \) is the imaginary unit. Expressing the exponential functions in terms of their trigonometric and hyperbolic equivalents and setting the entire imaginary part to zero, Eq. (3.27) can be re-written in the form

\[
\phi(x) = D_1 \cos \delta x + D_2 \sin \delta x + D_3 \cosh \varepsilon x + D_4 \sinh \varepsilon x
\]  

(3.29)

in which \( \delta = \imath s_3 \) and \( \varepsilon = s_1 \)

\[
\delta = \sqrt{\frac{T}{2EI}} \left(1 + \frac{4 \omega^2 \bar{m} EI}{T^2} - 1\right)
\]  

(3.30)

\[
\varepsilon = \sqrt{\frac{T}{2EI}} \left(1 + \frac{4 \omega^2 \bar{m} EI}{T^2} + 1\right)
\]  

(3.31)

The coefficients \( D_1, D_2, D_3 \) and \( D_4 \) can be evaluated by exactly the same procedure presented for the system without axial force.

### 3.3.2.1 Pinned-Pinned End Condition

Considering the simply-supported tie-rod with uniform section, subjected to a constant axial tensile force, its four boundary conditions are the same as those of Eqs. (a1) and (b1). Therefore
\[
\phi(0) = D_1 \cos 0 + D_2 \sin 0 + D_3 \cosh 0 + D_4 \sinh 0 = 0
\]
\[
\phi^\prime(0) = -\delta^2 D_1 \cos 0 - \delta^2 D_2 \sin 0 + \varepsilon^2 D_3 \cosh 0 + \varepsilon^2 D_4 \sinh 0 = 0
\]
These two equations give \( D_1 = D_3 = 0 \).
\[
\phi(L) = D_2 \sin \delta L + D_4 \sinh \varepsilon L = 0
\]
\[
\phi^\prime(L) = -\delta^2 D_2 \sin \delta L + \varepsilon^2 D_4 \sinh \varepsilon L = 0
\]
Adding these two equations and replacing \( D_4 \) by an expression of \( D_2 \). Only \( D_2 \) remains as a nonzero constant
\[
\phi(x) = D_2 \sin \delta x (1 + \frac{\delta^2}{\varepsilon^2})
\]
Excluding the trivial solution \( D_2 = 0 \), boundary condition \( \phi(L) = 0 \) can be satisfied only when
\[
\sin \delta L = 0 \Rightarrow \delta = n \pi / L \quad n = 0, 1, 2, ...
\]
Substituting this expression into Eq. (3.30) yield the frequency expression
\[
\omega_n = n \pi \sqrt{\frac{n^2 \pi^2 E I}{m L^4} + \frac{T}{m L^2}} \quad \text{(rad)} \quad n = 1, 2, 3, ...
\]
or
\[
f_n = \frac{n}{2} \sqrt{\frac{n^2 \pi^2 E I}{m L^4} + \frac{T}{m L^2}} \quad \text{(Hz)} \quad n = 1, 2, 3, ...
\]
In this case, if considering the bending stiffness of the tie-rod is unknown, there are only two unknown variables (\( T \) and \( EI \)). Therefore, it is necessary to know at least two natural frequencies.

The corresponding vibration mode shapes are now given by Eq. (e4) upon substitution of Eq. (g4) for the frequency parameter \( \delta \) in the sine term.

### 3.3.2.2 Fixed-Pinned End Condition

Considering the tie-rod with uniform section fixed at one end and pinned at the other end and subjected to a constant axial tensile force, its four boundary conditions are the same as those of Eqs. (a2) and (b2). Therefore
\[\phi(0) = D_1 \cos \theta + D_2 \sin \theta + D_3 \cosh \theta + D_4 \sinh \theta = 0\]
\[\phi'(0) = -\delta D_1 \sin \theta + \varepsilon D_2 \cos \theta + \varepsilon D_3 \sinh \theta + \varepsilon D_4 \cosh \theta = 0\]
\[\phi(L) = D_1 \cos \delta L + D_2 \sin \delta L + D_3 \varepsilon L + D_4 \sinh \varepsilon L = 0\]
\[\phi''(L) = -\delta^2 D_1 \cos \delta L - \delta^2 D_2 \sin \delta L + \delta^2 D_3 \cosh \varepsilon L + \delta^2 D_4 \sinh \varepsilon L = 0\]

The first two of these equations give \( D_3 = -D_1 \) and \( D_4 = -\left(\delta / \varepsilon\right) D_2 \). Substituting these equalities into the last two equations, and placing the resulting expressions in matrix form, one obtains

\[
\begin{bmatrix}
(\varepsilon \cos \delta L - \varepsilon \cosh \varepsilon L) & (\varepsilon \sin \delta L - \delta \sinh \varepsilon L) \\
(-\delta^2 \cos \delta L - \delta^2 \cosh \varepsilon L) & (-\delta^2 \sin \delta L - \delta \sinh \varepsilon L)
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(d5)

For coefficients \( D_1 \) and \( D_2 \) to be nonzero, the determination of the matrix must be zero

\[
\delta^2 (1 - \varepsilon) \sin \delta L \cos \delta L + \varepsilon (\delta^2 + \varepsilon) \sin \delta L \cosh \varepsilon L - \delta (\delta^2 + \varepsilon^2) \cos \delta L \sinh \varepsilon L = 0
\]

(e5)

The solution of this equation provides the values of \( \alpha L \) which represent the frequencies of vibration of the tie-rod with fixed-pinned end condition, subjected to a constant axial tensile force (Figure 3.3a).

![Figures 3.3](a) Eq. (e5) (b) Eq. (e6)

Figure 3.3 – Relationship between \( \delta \) and \( \varepsilon \): (a) fixed-pinned end condition; (b) fixed-fixed end condition

### 3.3.2.3 Fixed-Fixed End Condition

Considering the tie-rod with uniform section fixed at two ends, subjected to a constant axial tensile force, its four boundary conditions are the same as those of Eqs. (a3) and (b3). Therefore
\[ \phi(0) = D_1 \cos 0 + D_2 \sin 0 + D_3 \cosh 0 + D_4 \sinh 0 = 0 \]
\[ \phi'(0) = -\delta D_1 \sin 0 + \delta D_2 \cos 0 + \varepsilon D_3 \sinh 0 + \varepsilon D_4 \cosh 0 = 0 \]
\[ \phi(L) = D_1 \cos \delta L + D_2 \sin \delta L + D_3 \cosh \varepsilon L + D_4 \sinh \varepsilon L = 0 \]
\[ \phi'(0) = -\delta D_1 \sin \delta L + \delta D_2 \cos \delta L + \varepsilon D_3 \sinh \varepsilon L + \varepsilon D_4 \cosh \varepsilon L = 0 \]

Similarly, placing the expressions in matrix form
\[
\begin{bmatrix}
(e \cos \delta L - \varepsilon \cosh \varepsilon L) & (e \sin \delta L - \delta \sinh \varepsilon L) \\
(-\delta \sin \delta L - \varepsilon \sinh \varepsilon L) & (\delta \cos \delta L - \delta \cosh \varepsilon L)
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The determinant of the square matrix must equal zero, thus giving the frequency equation
\[ 2 \delta \varepsilon (1 - \cos \delta L \cosh \varepsilon L) + (\delta^2 - \varepsilon^2) \sin \delta L \sinh \varepsilon L = 0 \]

The solution of this equation represents the frequencies of vibration of the tie-rod with fixed-fixed end condition, subjected to a constant axial tensile force (Figure 3.3b).

Figure 3.3a and Figure 3.3b show the plots of the two variables \( \delta \) and \( \varepsilon \) with respect to each other in order to satisfy Eqs. (e6) and (e6), respectively. The program Matlab (2006) was used, assuming the length to be 5 m. To find the frequency value, Eqs. (3.30) and (3.31) can be used when the relationship between \( \delta \) and \( \varepsilon \) is known.

Different colours are used to indicate the shapes of different curves. For each value of \( \delta \), there are one or more than one values of \( \varepsilon \) which satisfy the boundary condition equations and vise versa. However, no closed form solution could be found, therefore, there is no common frequency equation like in the case of pinned-pinned end condition.

### 3.3.2.4 Summary of Frequency Formula of Tie-Rods Including Axial-Force Effects with Three Types of Boundary Conditions

A summary of the frequency formula of tie-rods including axial-force effects is given in Table 3-4. However, no closed form solutions existe for fixed-pinned and fixed-fixed conditions.

<table>
<thead>
<tr>
<th>System</th>
<th>Natural frequency ( \omega_n ) (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-pinned</td>
<td>( \omega_n = n \pi \sqrt{\frac{n^2 \pi^2 EL}{m L^4} + \frac{T}{m L^2}} ) for ( n = 1, 2, 3, \ldots )</td>
</tr>
<tr>
<td>Fixed-pinned</td>
<td>(- (*\text{to satisfy Eq. (e6)}) ) for ( n = 1, 2, 3, \ldots )</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td>(- (*\text{to satisfy Eq. (e6)}) ) for ( n = 1, 2, 3, \ldots )</td>
</tr>
</tbody>
</table>
Chapter 4
Numerical Parametric Study

Abstract

In this chapter, the dynamic analysis of tie-rods of different characteristics was performed using a Finite Element (FE) program DIANA. In total, there were 432 case studies conducted. The results from FE analysis will be compared with the theoretical approach and with a linear elastic element program. Then, the effect of tensile stress on the modes of vibration will be discussed, and the factors that affect the frequency of tie-rods will be studied. The effects of different factors will be assessed both numerically using FE results and analytically based on the theoretical equation. Moreover, the maximum effect of each factor will be estimated. During the process, discussions about the construction of several standard charts to determine the tensile stress from the frequency of different tie-rods will also be given. Finally, the estimation of the tensile stress based on an equation and standard charts will be concluded.
4.1 Introduction

The dynamic analysis of a steel tie-rod was performed using the program DIANA (2008). The modal frequency and the first ten modes of vibration were determined via free vibration at different values of applied tensile stresses. The tie-rod is assumed to be a beam, with uniform cross-section, subjected to a constant axial force. Furthermore, it is assumed to be supported by rotational springs at both ends. The value of the rotational spring is chosen to be close to 0 for pinned connection and infinity for fixed connection. The material properties for the target structure are the mass density of 7850 kg/m$^3$, the elastic modulus of 210 GPa and the Poisson’s ratio of 0.30 (Eurocode 3, 2003). The model has 20 elements and 21 nodes.

Beside the applied tensile stress, the different variants that were considered for the tie-rod include the bending stiffness (the type and size of the cross-section), length and boundary condition. Table 4-1 shows the geometric characteristics of the parametric case studies conducted.

Table 4-1
Geometrical characteristics of different tie-rods in numerical parametric study

<table>
<thead>
<tr>
<th>Length, $L$ (m)</th>
<th>Circular cross-section</th>
<th>Rectangular cross-section</th>
<th>Ratio $d/L$ (m/m)</th>
<th>Ratio $h/L$ (m/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Diameter, $d$ (mm)</td>
<td>Height, $h$ (mm)</td>
<td>Width, $b$ (mm)</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>-</td>
<td>10</td>
<td>40</td>
<td>0.0040</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>12</td>
<td>40</td>
<td>0.0048</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>20</td>
<td>40</td>
<td>0.0080</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>30</td>
<td>40</td>
<td>0.0120</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>0.0160</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>10</td>
<td>40</td>
<td>0.0020</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>12</td>
<td>40</td>
<td>0.0024</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>20</td>
<td>40</td>
<td>0.0040</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>30</td>
<td>40</td>
<td>0.0060</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>0.0080</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>10</td>
<td>40</td>
<td>0.0010</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>12</td>
<td>40</td>
<td>0.0012</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>20</td>
<td>40</td>
<td>0.0020</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>30</td>
<td>40</td>
<td>0.0030</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>0.0040</td>
</tr>
</tbody>
</table>
For each set of the length and cross-section values, three boundary conditions (i.e. pinned-pinned, fixed-pinned and fixed-fixed) and eight values of assumed tensile stresses (i.e. from 0, 5, 10, 25, 50, 100 to 200 MPa) were considered, resulting in a number of 24 analysis models. In total, there are 432 models that were analyzed. To facilitate the analysis process of this high number of models, a calculation file was written in the program Matlab (2006). Matlab was used as a pre-and post-processor for DIANA. A schematic representation of the work flow between DIANA and Matlab is presented in Figure 4.1.

Figure 4.1 – Work flow between DIANA and Matlab file

To compare the results obtained from DIANA, another set of models was analyzed using the program SAP2000. However, only a limited number of SAP2000 models were created.

The purpose of running different numerical analyses was to analyze the influence of each parameter in the behavior of the tie-rod model. After that, the purpose is to construct several standard charts to represent the frequency values of a number of tie-rods at different applied stresses. The charts should be able to cover a wide range of different tie-rods and will provide convenience to the target users.
4.1.1 Discussion of the Definition of the Numerical Model

The end constraint in the numerical parametric study has rotational springs with unknown stiffness, based on the fact that ancient tie-rods are often inserted into the masonry walls and the boundary condition depends on many factors as mentioned in Chapter 2. In practice, when carrying out the dynamic tests of simulated systems, the engineer has the challenge of defining and setting up the structural model which can represent the real tie-rods. One of the simplest models is the one of the “pinned-pinned beam”. However, Lagomarsino and Calderini (2005) stated that, on one hand this simplest model presents the advantage of allowing theoretical closed form solutions of the inverse problem. On the other it yields rather approximate results, since it neglects the stiffness contribution provided by constraints, leading to an overrating of the axial tensile force.

The approach adopted in this parametric study is to analyze some representative extreme cases of the boundary conditions of tie-rod which are relevant from the structural point of view. They are pinned-pinned, fixed-pinned and fixed-fixed systems. The first advantage of the approach is that it simplifies the constructing of the numerical models as well as the setting up of the experimental models. Secondly, in the pinned-pinned system, a closed form solution of the direct and inverse problems exists. More importantly, the boundary condition of real tie-rods is between these extreme cases; therefore, the parametric study is able to cover a complete range of actual frequencies and vibration modes of real tie-rods. As a result, it also provides the complete range for the estimation of tensile force $T$.

In terms of accuracy assumption, the finite element modelling through DIANA and SAP2000 provides two sets of reference solutions, both of which are assumed as the right ones after an appropriate convergence analysis. They will be compared with each other and with the theoretical equations.

Beside the “ideal” extreme value of rotational stiffness of the boundary condition, the definition of the models turns out to be not perfectly realistic for another reason: the bending stiffness along the tie-rod is non-homogeneous. Nevertheless, such approximations appear to be acceptable in many real situations.
4.2 Results of the Numerical Model

4.2.1 Comparison of FE Results with Linear Beam Results and Theoretical Approach

To compare the results of DIANA with SAP2000 and theoretical equations, the tie-rod model with the following characteristics was chosen: 5 m in length, 0.04 x 0.04 m in cross-section and pinned at two ends. The applied tensile stresses are eight values as defined in the previous section. The first four values indicate the low values of the tensile stresses acting on the tie-rods at the beginning states, whereas the values from 50 to 200 MPa represent the later states where the tie-rods have to bear more horizontal loads from the masonry structures. The value of 200 MPa is quite high and considered to be close to the yield stress of the old iron as mentioned in Chapter 2.

Using the complete theoretical equation for frequency of tie-rod of pinned-pinned condition derived in Chapter 3:

\[ f_n = \frac{n^2 \pi^2 EI}{mL^4} + \frac{T}{mL^2}, \]  

where \( f_n \) is the frequency of mode \( n^{\text{th}} \) (Hz); \( \omega_n \) is the circular frequency of mode \( n^{\text{th}} \) (rad); \( n \) is the mode number; \( L \) is the length of the tie-rod (m); \( EI \) is the bending stiffness (Nm²), \( T \) is the tensile force (N) and \( m \) is mass per unit length of the tie-rod (kg/m).

The results of the comparison are shown in Table 4-2 and Table 4-3.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( f ) (Hz) – ( \sigma = 0 ) (MPa)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DIANA</td>
<td>SAP2000</td>
</tr>
<tr>
<td>1</td>
<td>3.75</td>
<td>3.75</td>
</tr>
<tr>
<td>2</td>
<td>15.01</td>
<td>15.01</td>
</tr>
<tr>
<td>3</td>
<td>33.81</td>
<td>33.75</td>
</tr>
<tr>
<td>4</td>
<td>60.32</td>
<td>59.96</td>
</tr>
<tr>
<td>5</td>
<td>94.97</td>
<td>93.59</td>
</tr>
<tr>
<td>6</td>
<td>138.61</td>
<td>134.58</td>
</tr>
<tr>
<td>7</td>
<td>192.74</td>
<td>182.83</td>
</tr>
</tbody>
</table>
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Table 4-3
The frequency of the first ten modes: DIANA results vs. SAP2000 and theory – \( \sigma = 25 \) MPa

<table>
<thead>
<tr>
<th>Mode</th>
<th>( f (\text{Hz}) - \sigma = 25 ) (MPa)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DIANA</td>
<td>SAP2000</td>
</tr>
<tr>
<td>1</td>
<td>6.78</td>
<td>6.78</td>
</tr>
<tr>
<td>2</td>
<td>18.78</td>
<td>18.78</td>
</tr>
<tr>
<td>3</td>
<td>37.82</td>
<td>37.76</td>
</tr>
<tr>
<td>4</td>
<td>64.43</td>
<td>64.06</td>
</tr>
<tr>
<td>5</td>
<td>99.13</td>
<td>97.74</td>
</tr>
<tr>
<td>6</td>
<td>142.80</td>
<td>138.76</td>
</tr>
<tr>
<td>7</td>
<td>196.96</td>
<td>187.02</td>
</tr>
</tbody>
</table>

It can be seen that, the level of agreement between DIANA results of the first seven modes and SAP2000 or the theoretical approach is high. The DIANA results of the first six modes should be used for the parametric study. Further verification for the DIANA results of the first six modes is made taking into account the length effect (Table 4-4).

Table 4-4
The frequency at different tensile stresses and with three different lengths: DIANA vs. theoretical approach

<table>
<thead>
<tr>
<th>( \sigma ) (MPa)</th>
<th>( f_n ) (Hz) – Mode 1</th>
<th></th>
<th></th>
<th>( f_n ) (Hz) – Mode 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DIANA</td>
<td>SAP2000</td>
<td>Theory</td>
<td>DIANA</td>
</tr>
<tr>
<td>0</td>
<td>15</td>
<td>3.75</td>
<td>0.94</td>
<td>15.01</td>
</tr>
<tr>
<td>5</td>
<td>15.83</td>
<td>4.52</td>
<td>1.57</td>
<td>15.84</td>
</tr>
<tr>
<td>10</td>
<td>16.62</td>
<td>5.18</td>
<td>2.02</td>
<td>16.62</td>
</tr>
<tr>
<td>25</td>
<td>18.78</td>
<td>6.78</td>
<td>2.97</td>
<td>18.78</td>
</tr>
<tr>
<td>50</td>
<td>21.91</td>
<td>8.82</td>
<td>4.1</td>
<td>21.91</td>
</tr>
<tr>
<td>100</td>
<td>27.11</td>
<td>11.89</td>
<td>5.72</td>
<td>27.11</td>
</tr>
<tr>
<td>150</td>
<td>31.46</td>
<td>14.32</td>
<td>6.98</td>
<td>31.46</td>
</tr>
<tr>
<td>200</td>
<td>35.27</td>
<td>16.4</td>
<td>8.04</td>
<td>35.28</td>
</tr>
</tbody>
</table>

The results show that the frequencies for the first six modes obtained by DIANA and those calculated by theoretical formula match well. The agreement does not depend on the length or the value of applied tensile stress, although the results are closer for lower modes. It is possible to conclude that the results obtained from DIANA for the first six modes of rectangular tie-rods with pinned-pinned end condition are proved to be reliable and fit well with the theoretical Eq. (4.1).
4.2.2 Effect of Tensile Stress on the Modes of Vibration of Tie-Rod

Figure 4.2, Figure 4.3 and Figure 4.4 present the results of the first six modes of vibration obtained by DIANA at eight levels of applied tensile stresses, considering three different lengths and boundary conditions.

Figure 4.2 – The first six modes at different tensile stresses – tie-rod of 2.5 m long and 0.04x 0.04 m square
Figure 4.3 – The first six modes at different tensile stresses – tie-rod of 5 m long and 0.04 x 0.04 m square
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$L = 10 \text{ m}$

<table>
<thead>
<tr>
<th>Pinned-pinned</th>
<th>Fixed-pinned</th>
<th>Fixed-fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1&lt;sup&gt;st&lt;/sup&gt; mode</td>
<td>1&lt;sup&gt;st&lt;/sup&gt; mode</td>
</tr>
<tr>
<td></td>
<td>2&lt;sup&gt;nd&lt;/sup&gt; mode</td>
<td>2&lt;sup&gt;nd&lt;/sup&gt; mode</td>
</tr>
<tr>
<td></td>
<td>3&lt;sup&gt;rd&lt;/sup&gt; mode</td>
<td>3&lt;sup&gt;rd&lt;/sup&gt; mode</td>
</tr>
<tr>
<td></td>
<td>4&lt;sup&gt;th&lt;/sup&gt; mode</td>
<td>4&lt;sup&gt;th&lt;/sup&gt; mode</td>
</tr>
<tr>
<td></td>
<td>5&lt;sup&gt;th&lt;/sup&gt; mode</td>
<td>5&lt;sup&gt;th&lt;/sup&gt; mode</td>
</tr>
<tr>
<td></td>
<td>6&lt;sup&gt;th&lt;/sup&gt; mode</td>
<td>6&lt;sup&gt;th&lt;/sup&gt; mode</td>
</tr>
</tbody>
</table>

Figure 4.4 – The first six modes at different tensile stresses – tie-rod of 10 m long and 0.04 x 0.04 m square
The results of the mode shapes as shown above are available from two scenarios, e.g. zero applied tensile stress and non-zero applied tensile stress stages. To study the numerical correlation between two sets of mode shape vectors, the MAC and COMAC values will be calculated.

### 4.2.2.1 Modal Assurance Criteria (MAC)

MAC value is given by:

\[
MAC_{(i,j)} = \frac{\sum_{j=1}^{n} \varphi_{iz}^T \cdot \varphi_{in}}{\sum_{i=1}^{n} (\varphi_{iz}^T \cdot \varphi_{iz})(\varphi_{in}^T \cdot \varphi_{in})}
\]

where \(\varphi_{iz}\) and \(\varphi_{in}\) are the mode shape vectors of the \(i^{th}\) mode in zero applied tensile stress and non-zero applied tensile stress stages, the superscript \(T\) is transposed vector and \(n\) indicates the number of estimated degree of freedom. The MAC takes on values from zero – representing no consistent correspondence, to one – representing a consistent correspondence.

Figure 4.5 presents the results of the MAC values for the first six modes at eight levels of applied tensile stress. The reference mode shape vector used is the vector in zero applied tensile stress. Although the level of conformity is very high as all the MAC values are above 0.95, the effect of tensile stress on the vibration modes can be seen. The magnitude of effect depends on the length, the mode, the boundary condition and value of applied tensile stress.

### 4.2.2.2 Co-ordinate Modal Assurance Criteria (COMAC)

The Co-ordinate Modal Assurance Criterion (COMAC) is an extension of MAC. When searching for local information, the COMAC can be used by the following expression:

\[
COMAC_{(j)} = \frac{\sum_{j=1}^{m} \varphi_{iz,j} \cdot \varphi_{in,j}}{\sum_{j=1}^{m} (\varphi_{iz,j})^2 \cdot \sum_{j=1}^{m} (\varphi_{in,j})^2}
\]

where \(\varphi_{iz,j}\) and \(\varphi_{in,j}\) are the displacement in points \(j\) during the vibrations in the \(i^{th}\) mode in zero applied tensile stress and non-zero applied tensile stress stages, respectively, and \(m\) is the number of estimated mode shapes. The COMAC attempts to identify which degrees of freedom contribute negatively to a low value of MAC. Figure 4.6 shows the COMAC values for the first mode of vibration of tie-rods. The reference set of mode shape vectors consists of six vectors of the first six modes at zero applied tensile stress.
The figure shows the MAC values for the first six modes at different applied tensile stresses for three types of boundary conditions: pinned-pinned, fixed-pinned, and fixed-fixed. The lengths of the tie-rods are 2.5 m, 5 m, and 10 m for each condition.

Figure 4.5 – MAC values for the first six modes at different applied tensile stresses

The second figure shows the COMAC values for the first mode at different tensile stresses for the same boundary conditions.

Figure 4.6 – COMAC values for the first mode at different tensile stresses
From the MAC and COMAC values, some conclusions include: the effect of tensile stress is more significant when the boundary condition is fixed-fixed and fixed-pinned compared to pinned-pinned condition; the effect of tensile stress is more significant for the lower modes, especially for the 1\textsuperscript{st}, 2\textsuperscript{nd} and 3\textsuperscript{rd} modes and reduced when the mode is getting higher; and the effect of tensile stress is more significant when the length is increased.

4.3 Factors Affecting the Frequency of Tie-Rod

The results of the DIANA models are presented in Figure 4.7, as tensile stress versus frequency value, for the first mode.

It can be seen in Figure 4.7 that

- The effect of tensile stress is that: the higher the applied tensile stress, the higher the frequency;
- The effect of boundary condition is that: the more restrained the boundary condition, the higher the frequency;
- The effect of length is that: the longer the tie-rod, the lower the frequency;
- The effect of size of cross-section is that: the bigger the cross-section, the higher the frequency;
- The effect of shape of cross-section is that: the shape affects the response of the frequency at different applied tensile stress. When the shape is the same, the frequency response at different applied tensile follows a similar trend; when the shape is different and the size is the same, the change in the frequency of circular tie-rods due to different applied tensile stress is less than the rectangular tie-rods. This means the circular cross-section is less sensitive to the applied tensile stress than the rectangular one;
- The effect of cross-section in combination with the effect of length is that: the longer the length, the less significant the effect of cross-section. The length effect is more significant than the bending stiffness effect.
- The effect of length in combination with the effect of the boundary condition is that: the length effect is more significant than the boundary condition effect.
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Figure 4.7 – DIANA results of all case studies conducted
### 4.3.1 Effect of Tensile Stress

Figure 4.8 shows the frequencies of the first six modes with two levels of tensile stresses: 0 and 200 MPa. It can be seen that the tensile stress increases the frequency of all modes and the magnitude of difference depends on the length, the boundary condition and the order of the mode.

Based on the frequency values, because the lower modes have much lower frequency values than the higher modes, the effect of tensile stress is much more significant on the lower modes. In addition, the effect is more significant when the length is increased.
The frequency ratios are calculated by dividing the frequency at higher tensile stress by that at zero tensile stress. The results are given in Table 4-5. These are the maximum ratios selected from all the case studies, which are for the 10m-long tie-rod with pinned-pinned condition. The ratios between different case studies are not consistent.

Table 4-5
Maximum frequency ratios due to the effect of tensile stress

<table>
<thead>
<tr>
<th>Mode</th>
<th>Max. Frequency Ratio: ( f_\sigma / f_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress (MPa)</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>1.000</td>
</tr>
<tr>
<td>3</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
</tr>
<tr>
<td>5</td>
<td>1.000</td>
</tr>
<tr>
<td>6</td>
<td>1.000</td>
</tr>
</tbody>
</table>

### 4.3.2 Effect of Boundary Condition

The frequency is increased when the boundary condition is fixed-fixed compared to pinned-pinned. The amount of increase depends on the length, the cross-section and the tensile stress. When the tensile stresses are small, the frequency of the tie-rod is more sensitive to the boundary condition. In addition, the shorter the tie-rod, the more significant and less linear the increase in the frequency. In summary, the changes in the frequency values due to higher tensile stresses are more significant for shorter tie-rosds and pinned-pinned condition.

To find the ratios of the frequency between different boundary conditions, the frequency values of the fixed-pinned and fixed-fixed condition are divided by those of pinned-pinned condition for all the case studies conducted. There are a total of 288 case studies, as a result of 6 cross-sections, 3 lengths, 2 boundary conditions and 8 values of tensile stresses. For each case study, all other variables are kept the same except the boundary condition. To be conservative, maximum ratios are chosen and presented in Table 4-6 and Table 4-7.

The ratios obtained are highly consistent regardless of the effects of the bending stiffness and length, proving that these ratios can be used as approximation factors to estimate the frequency when only the boundary condition of the tie-rod is changed. The maximum ratio is 2.267, meaning that the frequency of a tie-rod with arbitrary end connection is in the range from 1.0 to 2.267 times the frequency of a tie-rod with pinned-pinned supports.
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Table 4-6
Frequency ratio between fixed-pinned and pinned-pinned condition

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency Ratio: Fixed-pinned/ pinned-pinned</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress (MPa)</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.562</td>
</tr>
<tr>
<td>2</td>
<td>1.266</td>
</tr>
<tr>
<td>3</td>
<td>1.174</td>
</tr>
<tr>
<td>4</td>
<td>1.131</td>
</tr>
<tr>
<td>5</td>
<td>1.106</td>
</tr>
<tr>
<td>6</td>
<td>1.090</td>
</tr>
</tbody>
</table>

Table 4-7
Frequency ratio between fixed-fixed and pinned-pinned condition

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency Ratio: Fixed-fixed/ pinned-pinned</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress (MPa)</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2.267</td>
</tr>
<tr>
<td>2</td>
<td>1.563</td>
</tr>
<tr>
<td>3</td>
<td>1.363</td>
</tr>
<tr>
<td>4</td>
<td>1.270</td>
</tr>
<tr>
<td>5</td>
<td>1.217</td>
</tr>
<tr>
<td>6</td>
<td>1.185</td>
</tr>
</tbody>
</table>

### 4.3.3 Effect of Bending Stiffness

Table 4-8 shows the frequency ratios between the 0.04x0.04 m cross-section divided by the frequency of the 0.01x0.04 m cross-section (i.e. stiffness ratio is equal to 4.0). These are maximum ratios selected among the case studies. The ratios between different case studies are not consistent. The maximum effect that could be seen is that when the height of the cross-section is increased $x_h$ times, the frequency is increased $x_h$ times.

Table 4-8
Frequency ratio – effect of the bending stiffness

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency ratio – Height ratio of rectangular cross-section = 4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress (MPa)</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3.996</td>
</tr>
<tr>
<td>2</td>
<td>3.993</td>
</tr>
<tr>
<td>3</td>
<td>3.985</td>
</tr>
<tr>
<td>4</td>
<td>3.974</td>
</tr>
<tr>
<td>5</td>
<td>3.959</td>
</tr>
</tbody>
</table>
4.3.4 Effect of Length

The frequency ratios are calculated for tie-rods of different lengths. Choosing a fixed length ratio of 2.0, the following tables demonstrate the effect of other factors in combination with the length effect, proving that the length effect is relatively complicated and dependent on the effect of other factors.

So unlike the boundary condition effect, the length effect must be assessed in combination with other effects. It can be seen that the length effect is significant and negative; maximum effect is that when increasing the length \(x_L\) times, the frequency is reduced \(x_L^2\) times.

Table 4-9
Frequency ratio of first mode between tie-rods of different lengths

<table>
<thead>
<tr>
<th>Length</th>
<th>Frequency ratio – Length ratio = 2.0</th>
<th>Tensile stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 m / 2.5 m</td>
<td></td>
<td>0 5 10 25 50 100 150 200</td>
</tr>
<tr>
<td>10 m / 5 m</td>
<td></td>
<td>0.250 0.286 0.312 0.361 0.403 0.439 0.455 0.465</td>
</tr>
</tbody>
</table>

Table 4-10
Frequency ratio of first mode – length effect in combination with bending stiffness effect

<table>
<thead>
<tr>
<th>Cross-section</th>
<th>Frequency ratio – Length ratio = 2.0</th>
<th>Tensile stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 5 10 25 50 100 150 200</td>
</tr>
<tr>
<td>0.04x0.04 m</td>
<td></td>
<td>0.250 0.286 0.312 0.361 0.403 0.439 0.455 0.465</td>
</tr>
<tr>
<td>0.01x0.04 m</td>
<td></td>
<td>0.250 0.428 0.458 0.481 0.490 0.495 0.497 0.497</td>
</tr>
</tbody>
</table>

Table 4-11
Frequency ratio of first mode – length effect in combination with boundary condition effect

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Frequency ratio – Length ratio = 2.0</th>
<th>Tensile stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 5 10 25 50 100 150 200</td>
</tr>
<tr>
<td>Pinned-pinned</td>
<td></td>
<td>0.250 0.286 0.312 0.361 0.403 0.439 0.455 0.465</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td></td>
<td>0.250 0.260 0.269 0.292 0.319 0.355 0.378 0.394</td>
</tr>
</tbody>
</table>

Table 4-12
Frequency ratio – length effect in combination with mode order effect

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency ratio – Length ratio = 2.0</th>
<th>Tensile stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 5 10 25 50 100 150 200</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.250 0.286 0.312 0.361 0.403 0.439 0.455 0.465</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.250 0.260 0.270 0.293 0.323 0.361 0.386 0.403</td>
</tr>
</tbody>
</table>
4.3.5 Effect of the Relation between Mode

The mode has a significant effect on the frequency value. Table 4-13 shows the frequency ratios of the frequencies of the second to sixth modes divided by the frequency of the first mode, these are maximum ratios selected among the case studies. The ratios between different case studies are not consistent, meaning that the effect of mode order also depends on other factors’ effects. The maximum effect of the order of the mode is that when the mode is increased $x_{mode}$ times, the frequency is increased by $x_{mode}^2$ times.

Table 4-13
Frequency ratio – effect of the mode order

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency ratio: $n^{th}$ mode/ 1$^{st}$ mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress (MPa)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3.996</td>
</tr>
<tr>
<td>5</td>
<td>25.113</td>
</tr>
<tr>
<td>6</td>
<td>36.520</td>
</tr>
</tbody>
</table>

4.3.6 Summary of the Effect of the Factors Affecting the Frequency of Tie-Rods

Table 4-14 summarizes the effect of all the factors that affect the frequency of tie-rods obtained from the results of 432 models in DIANA. The maximum effect in the table is estimated based on numerical results without analytical verification. The analytical verification will be discussed in the next Section.

Table 4-14
Factors affecting the frequency of tie-rods based on numerical results

<table>
<thead>
<tr>
<th>No.</th>
<th>Factor</th>
<th>Effect*</th>
<th>Estimated maximum effect</th>
<th>Level of influence*</th>
<th>Level of dependence*</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>Tensile stress</td>
<td>(+)</td>
<td>8.566</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>02</td>
<td>Boundary condition</td>
<td>(+)</td>
<td>2.267</td>
<td>Medium</td>
<td>Low</td>
</tr>
<tr>
<td>03</td>
<td>Size of cross-section</td>
<td>(+)</td>
<td>$x_h$</td>
<td>Medium</td>
<td>High</td>
</tr>
<tr>
<td>04</td>
<td>Mode number</td>
<td>(+)</td>
<td>$(x_{mode})^2$</td>
<td>High</td>
<td>High</td>
</tr>
<tr>
<td>05</td>
<td>Length</td>
<td>(−)</td>
<td>$1/xL^2$</td>
<td>High</td>
<td>High</td>
</tr>
<tr>
<td>06</td>
<td>Shape of cross-section</td>
<td>(−)</td>
<td>unknown</td>
<td>Medium</td>
<td>High</td>
</tr>
</tbody>
</table>
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

*Remark in Table 4-14:

- Effect: (+) means positive effect, such that when the factor is increased, the frequency is also increased and vice versa. This is opposite to (–) or negative effect.
- “Level of influence”: the effect on the frequency in comparison with other factors’ effects.
- “Level of dependence”: the change of the effect on the frequency in combination with other factors’ effects.
- The classification of level of influence and level of dependence is based on the Author’s judgement and solely for the convenience of interpreting the results.

4.4 Study of the Factors Affecting the Frequency of Tie-Rod Based on Theoretical Equations

The purpose of this section is to verify analytically the results obtained in DIANA of the factors affecting the frequency of tie-rod. First, the effect of each factor will be assessed based on theoretical equations to verify the estimated maximum effect in Table 4-14. Furthermore, a study of some correlation factors to combine the different effects will be carried out. During the process, discussions about the methodology to construct several standard charts will be given.

The results from DIANA program for the first six modes are used for the parametric study, and they have proved to comply well with the theoretical Eq. (4.1). At this point, it is assumed that the Eq. (4.1) closely represents the behavior of a real steel tie-rod with pinned-pinned end condition. The accuracy level of this equation will be assessed by comparing with the experimental results in the later Chapters.

Based on Eqs. (4.10) and (4.12) derived below, assuming $E$ and $\rho$ are constant, the factors that affect the frequency of the tie-rod based on the theoretical equations are: the mode number ($n$), the length of the tie-rod ($L$), the tensile stress ($\sigma$) and the height of the rectangular cross-section ($h$), or the diameter of the circular cross-section ($d$). Note that the width ($b$) of the rectangular cross-section imposes no effect on the frequency of the tie-rod based on the Eq. (4.11), which is likely not accurate in real cases. However, it is reasonable that $h$ contributes more significantly compared to $b$. On the other hand, the difference between the rectangular and circular cross-sections, if $h = d$, results from the constants “1/12” and “1/16” in the first term inside the square root of Eqs. (4.11) and (4.13).
### Rectangular cross-section

- **Moment of Inertia (m⁴):**
  \[ I_o = \frac{b \times h^3}{12} \]  
  (4.4)
  where, \( b \): width of the cross-section (m);
  \( h \): height of the cross-section (m).

- **Mass per unit length (kg/m):**
  \[ m_o = \rho \times b \times h \]  
  (4.6)
  where, \( \rho \): mass density of tie-rod (kg/m³).

- **Tensile force (N):**
  \[ T_o = \sigma \times b \times h \]  
  (4.8)
  where, \( \sigma \): tensile stress (N/m²).

- **Frequency of rectangular tie-rod (\( f_{m_o} \)), pinned at two ends:** by substituting Eqs. (4.4), (4.6) and (4.8) into Eq. (4.1):
  \[ f_{m_o} = \frac{n}{2} \sqrt{\frac{n^2 \pi^2 E \times h^2}{12 \times \rho_s \times L^4} + \frac{\sigma}{\rho_s \times L^2}} \]  
  (4.10)

### Circular cross-section

- **Moment of Inertia (m⁴):**
  \[ I_o = \frac{\pi \times r^4}{4} \]  
  (4.5)
  where, \( r \): radius of the cross-section (m).

- **Mass per unit length (kg/m):**
  \[ m_o = \rho \times \pi \times r^2 \]  
  (4.7)

- **Tensile force (N):**
  \[ T_o = \sigma \times \pi \times r^2 \]  
  (4.9)

- **Frequency of circular tie-rod (\( f_{m_o} \)), pinned at two ends:** by substituting Eqs. (4.5), (4.7) and (4.9) into Eq. (4.1):
  \[ f_{m_o} = \frac{n}{2} \sqrt{\frac{n^2 \pi^2 E \times r^2}{4 \times \rho_s \times L^4} + \frac{\sigma}{\rho_s \times L^2}} \]  
  (4.12)

- **Frequency of circular tie-rod (\( f_{m_o} \)), pinned at two ends:** by substituting Eqs. (4.5), (4.7) and (4.9) into Eq. (4.1):
  \[ f_{m_o} = \frac{n}{2} \sqrt{\frac{n^2 \pi^2 E \times d^2}{16 \times L^4} + \frac{\sigma}{\rho_s \times L^2}} \]  
  (4.13)

where, \( d \): diameter of the cross-section (m).
### 4.4.1 Effect of Bending Stiffness

#### 4.4.1.1 Effect of Shape of Cross-Section

The two types of cross-sections of ancient tie-rods are rectangular and circular. The purpose of this Section is to find a factor relating the frequency of a rectangular tie-rod to the frequency of a circular one for which \( d = h \). All other parameters other than \( h \) or \( d \) are assumed to be the same and there are pinned-pinned end conditions.

<table>
<thead>
<tr>
<th>Rectangular cross-section</th>
<th>Circular cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{h}{L} )</td>
<td>( \frac{r}{L} )</td>
</tr>
<tr>
<td>A - A</td>
<td>A - A</td>
</tr>
</tbody>
</table>

Relation between \( f_{nc} \) and \( f_{no} \) (\( h = d \)):

\[
f_{nc} = \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \sqrt{\frac{4}{3} \times \left( \frac{2\sqrt{\rho_s} \times \frac{n}{L}}{n} \right) \times f_{no}^2 - \frac{1}{3} \times \sigma}
\]

\[
f_{nc} = \sqrt{\frac{4}{3} \times f_{no}^2 - \frac{1}{3} \times \left( \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \right)^2} \times \sigma \quad (4.14)
\]

Using \( \rho_s \) is 7850 kg/m\(^3\):

substituting into equation (4.14):

\[
f_{nc} = \sqrt{1.333 \times f_{no}^2 - 10.62 \times 10^{-6} \left( \frac{n}{L} \right)^2} \times \sigma \quad (4.16)
\]

Relation between \( f_{no} \) and \( f_{nc} \) (\( d = h \)):

\[
f_{no} = \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \sqrt{\frac{3}{4} \times \left( \frac{2\sqrt{\rho_s} \times \frac{n}{L}}{n} \right) \times f_{nc}^2 + \frac{1}{4} \times \sigma}
\]

\[
f_{no} = \sqrt{\frac{3}{4} \times f_{nc}^2 + \frac{1}{4} \times \left( \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \right)^2} \times \sigma \quad (4.15)
\]

Using \( \rho_s \) is 7850 kg/m\(^3\):

substituting into equation (4.15):

\[
f_{no} = \sqrt{0.75 \times f_{nc}^2 + 7.96 \times 10^{-6} \left( \frac{n}{L} \right)^2} \times \sigma \quad (4.17)
\]

Eq. (4.14) or (4.15) represents the general relationship between \( f_{nc} \) and \( f_{no} \) when \( h = d \). The relation is not straightforward, no constant factor can be found.

From the equations, the frequencies of circular tie-rods are always smaller than those of rectangular tie-rods when \( d = h \). Generally, either Eq. (4.14) or (4.15) can be used to calculate \( f_{no} \) when \( f_{nc} \) is known. When \( \rho_s \) is 7850 kg/m\(^3\), either Eq. (4.16) or (4.17) can be used.
### 4.4.1.2 Effect of Size of Cross-Section

The purpose of this Section is to determine how much the frequency change when the size of the cross-section changes.

Let \( f_n \) be the frequency of a tie-rod at mode \( n \), having height \( h \) (or diameter \( d \)), length \( L \) and tensile stress \( \sigma \). This tie-rod is called the “original” tie-rod.

#### Rectangular cross-section

<table>
<thead>
<tr>
<th>Expression</th>
<th>Circular cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Original” tie-rod: ( f_{n, o} = \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \sqrt{\frac{n^2 \pi^2 E \times h^2}{12 \times L^2} + \sigma} )</td>
<td>“Original” tie-rod: ( f_{n, o} = \frac{1}{2\sqrt{\rho_s}} \times \frac{n}{L} \sqrt{\frac{n^2 \pi^2 E \times d^2}{16 \times L^2} + \sigma} )</td>
</tr>
</tbody>
</table>

When \( n, h, L \) or \( \sigma \) are changed:

\[
\frac{f'_{n, o}}{f_{n, o}} = \frac{n'}{n} \times \frac{L}{L'} \sqrt{\frac{n^2 \pi^2 E \times h'^2}{12 \times L'^2} + \sigma'}
\]

To study the effect of all variables \( n, L, \sigma \) and \( h \) or \( d \), let \( x_{f,n} = \frac{f'_{n}}{f_{n}} \), \( x_{mode} = \frac{n'}{n} \), \( x_{L} = \frac{L'}{L} \):

\[
x_{h} = \frac{h'}{h}, \quad x_{d} = \frac{d'}{d}, \quad x_{\sigma} = \frac{\sigma'}{\sigma}
\]

\[
x_{f,n} = \frac{x_{mode}}{x_{L}} \times \sqrt{\left(\frac{x_{f,n}}{x_{L}}\right)^2 \frac{n^2 \pi^2 E \times h^2}{12 \times L^2} + x_{\sigma} \sigma}
\]

For rectangular cross-section:

\[
x_{f,n} = \frac{x_{mode}}{x_{L}} \times \sqrt{\left(\frac{x_{mode}}{x_{L}}\right)^2 \frac{1}{\frac{n^2 \pi^2 E \times h^2}{12 \times L^2} + \sigma} + \frac{\sigma}{\frac{n^2 \pi^2 E \times h^2}{12 \times L^2} + \sigma}}
\]
Similarly for circular cross-sections; introducing $k_n$ and $k_o$:

<table>
<thead>
<tr>
<th>Rectangular cross-section</th>
<th>Circular cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $k_n = \frac{\sigma}{\left(\frac{n^2 \pi^2 E \times h^2}{12 \times L^2} + \sigma\right)}$ (4.18)</td>
<td>Let $k_o = \frac{\sigma}{\left(\frac{n^2 \pi^2 E \times d^2}{16 \times L^2} + \sigma\right)}$ (4.19)</td>
</tr>
</tbody>
</table>

\[
x_f = x_{\text{mode}} \frac{x_L^2 \left(1 - k_n\right) + x_s k_n}{x_L^2} \text{ (4.20)}
\]

\[
x_{fo} = x_{\text{mode}} \frac{x_L^2 \left(1 - k_o\right) + x_s k_o}{x_L^2} \text{ (4.21)}
\]

Eqs. (4.20) and (4.21) represent the relationship as a multiplying factor between the new frequency and frequency of the “original” tie-rod.

The tie-rod with the characteristics presented in Table 4-11 was chosen as the “original” tie-rod for further study.

Table 4-15
Parameters of the “original” tie-rod (rectangular or circular cross-section) used in parametric study

<table>
<thead>
<tr>
<th>L (m)</th>
<th>h (m)</th>
<th>b (m)</th>
<th>d (m)</th>
<th>$\sigma$ (N/m²)</th>
<th>n</th>
<th>$E$ (N/m²)</th>
<th>$\rho_s$ (kg/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>5 x10⁶</td>
<td>1</td>
<td>2.1 x10¹¹</td>
<td>7850</td>
</tr>
</tbody>
</table>

Substituting the parameter values of the “original” tie-rod into the equations (4.11) and (4.13) for $f_n$ and $f_o$, as well as the expressions (4.18) and (4.19) for $k_n$ and $k_o$, one obtains

<table>
<thead>
<tr>
<th>Rectangular cross-section</th>
<th>Circular cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Original” tie-rod:</td>
<td>“Original” tie-rod:</td>
</tr>
<tr>
<td>$\Rightarrow f_{n} = 4.525$ (Hz) (4.22)</td>
<td>$\Rightarrow f_{o} = 4.119$ (Hz) (4.23)</td>
</tr>
<tr>
<td>$\Rightarrow k_n = 0.31145$ (4.24)</td>
<td>$\Rightarrow k_o = 0.37621$ (4.25)</td>
</tr>
</tbody>
</table>

To double-check, the values of $f_{n} = 4.525$ (Hz) and $f_{o} = 4.119$ (Hz) also match perfectly with the results from DIANA and SAP2000 programs.
When only the height (or the diameter) of the “original” tie-rod is changed:

If $\alpha_{mode} = \alpha_L = \alpha_\sigma = 1$:

$$x_{f_{in}} = \sqrt{(1-k_o)x_h^2 + k_o}$$  \hspace{1cm} (4.26)$$

$$\Rightarrow x_{f_{in}} = \sqrt{0.68855x_h^2 + 0.31145}$$

If $\alpha_{mode} = \alpha_L = \alpha_\sigma = 1$:

$$x_{f_{no}} = \sqrt{(1-k_o)x_d^2 + k_o}$$  \hspace{1cm} (4.27)$$

$$\Rightarrow x_{f_{no}} = \sqrt{0.62379x_d^2 + 0.37621}$$

To choose a range for $h$ (or $d$) provided that $h = d = 0.04$ m, assuming a wide range of $h'$ (or $d'$) such that: $0.004$ m $\leq h' = d' \leq 0.08$ m $\Rightarrow 0.1 \leq x_h = x_d = \frac{h'}{h} = \frac{d'}{d} = 0.04$ (m) $\leq 2.0$.

Eqs. (4.26) and (4.27) can be used to determine the new frequency based on the “original” frequency $f_{in}$ (or $f_{no}$) when only the height (or the diameter) of the “original” tie-rod is changed.

To combine with the effect of different tensile stresses (i.e. $\alpha_{mode} = \alpha_L = 1$ and $\alpha_\sigma \neq 1$ (or $x_d \neq 1$) and $\alpha_\sigma \neq 1$):

From equation (4.20):

if $\alpha_{mode} = \alpha_L = 1$:

$$x_{f_{in}} = \sqrt{(1-k_o)x_h^2 + k_o x_\sigma}$$  \hspace{1cm} (4.28)$$

$$\Rightarrow x_{f_{in}} = \sqrt{0.68855x_h^2 + 0.31145x_\sigma}$$

From equation (4.21):

if $\alpha_{mode} = \alpha_L = 1$:

$$x_{f_{no}} = \sqrt{(1-k_o)x_d^2 + k_o x_\sigma}$$  \hspace{1cm} (4.29)$$

$$\Rightarrow x_{f_{no}} = \sqrt{0.62379x_d^2 + 0.37621x_\sigma}$$

Choosing a range of $\sigma'$ such that: $0$ MPa $\leq \sigma' \leq 200$ MPa, being equivalent to a range of $x_\sigma$:

$0 \leq x_\sigma = \frac{\sigma'}{\sigma} = \frac{\sigma'}{5 \times 10^6}$ (N/m$^2$) $\leq 40$. The graphs of Eqs. (4.28) and (4.29) are shown in Figure 4.9.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Rectangular cross-section:
If \( x_{\text{mode}} = x_L = 1 \) \((n' = 1, L' = 5 \text{ m})\):

- \( \sigma = 0 \) MPa
- \( \sigma = 5 \) MPa
- \( \sigma = 50 \) MPa
- \( \sigma = 100 \) MPa

Circular cross-section:
If \( x_{\text{mode}} = x_L = 1 \) \((n' = 1, L' = 5 \text{ m})\):

- \( \sigma = 10 \) MPa
- \( \sigma = 25 \) MPa
- \( \sigma = 150 \) MPa
- \( \sigma = 200 \) MPa

Alternatively, the graphs of frequency ratios versus tensile stresses can be plotted as below:

Tie-rod of 5 m long – 1\( ^{\text{st}} \) mode

Figure 4.9 – Effect of different sizes of cross-section at different tensile stresses based on theoretical equation
4.4.1.3 Combined Effect of Shape and Size of Cross-Section

It is possible at this point to combine the effect of different sizes of cross-section of tie-rods with their shapes. The purpose is of this Section is to find some “correction factors” to relate the frequencies of circular tie-rod with that of rectangular one. Assume the circular tie-rod has the same characteristics as the rectangular tie-rod, and \( d = h \), \( x_{\text{mode}} = x_L = 1 \). Recalling Eq. (4.29):

\[
x_{f_{\text{no}}} = \sqrt{(1-k_o) x_d^2 + k_o x_{\sigma}} = \sqrt{(1-k_o) x_d^2 + k_o k_{\sigma} x_{\sigma}} = \sqrt{k_o (1-k_o) x_d^2 + k_o x_{\sigma}} - k_o \frac{k_o}{k_o - (1-k_o)}
\]

\[
\leftrightarrow x_{f_{\text{no}}} = \sqrt{k_o \frac{k_o (1-k_o)}{k_o - (1-k_o)}} x_d^2 + k_o x_{\sigma}
\]

\[
\leftrightarrow x_{f_{\text{no}}} = \sqrt{k_o \frac{k_o (1-k_o)}{k_o - (1-k_o)}} \left( x_d \right)^2 + k_o x_{\sigma}
\]

Let \( x_d = x_h \) \( \leftrightarrow x_d = \left( \frac{k_o (1-k_o)}{k_o - (1-k_o)} \right) x_h \):

\[
\rightarrow x_{f_{\text{no}}} = \sqrt{k_o \frac{k_o (1-k_o)}{k_o - (1-k_o)}} \left( \frac{k_o (1-k_o)}{k_o - (1-k_o)} x_h \right)^2 + k_o x_{\sigma}
\]

\[
\rightarrow x_{f_{\text{no}}} = \sqrt{k_o \frac{k_o (1-k_o)}{k_o - (1-k_o)}} x_h^2 + k_o x_{\sigma} = \frac{k_o x_{f_{\text{no}}}(x_h)}{k_o} = x_{f_{\text{no}}}.'
\]

Comparing Eq. (4.30) with (4.28), we can see Eq. (4.30) suggests two “correction factors” that could be used to relate the frequency ratios of circular and rectangular tie-rods:

1. Correction factor 1: \( \text{CF}_{(\text{shape}_{-1})} = \frac{k_o (1-k_o)}{k_o - (1-k_o)} \) such that \( x_d = \text{CF}_{(\text{shape}_{-1})} x_h = x_h.' \)

2. Correction factor 2: \( \text{CF}_{(\text{shape}_{-2})} = \frac{k_o}{k_o - (1-k_o)} \) such that \( x_{f_{\text{no}}} = x_{f_{\text{no}}}(x_h) = \text{CF}_{(\text{shape}_{-2})} x_{f_{\text{no}}}(x_h) \).

To examine how the two corrections work and for more convenience, substituting the values of \( k_o = 0.31145 \) and \( k_o = 0.37621 \) into the expressions for \( \text{CF}_{(\text{shape}_{-1})} \) and \( \text{CF}_{(\text{shape}_{-2})} \), we obtain \( \text{CF}_{(\text{shape}_{-1})} = 0.86603 \) and \( \text{CF}_{(\text{shape}_{-2})} = 1.0991 \).
The following steps show how to construct the graph of $x_{f_{no}}$ from the equation and graph of $x_{f_{ne}}$, recalling Eq. (4.28):

$$x_{f_{ne}} = \sqrt{(1-k_{c})x_{h}^{2} + k_{c}x_{\sigma}}$$

$$x_{f_{ne}} = \sqrt{0.68855x_{h}^{2} + 0.31145x_{\sigma}}$$

**Step 1**: Calculate $x_{h}' = CF_{(shape\_1)}x_{h}$

$$x_{h}' = 0.86603x_{h}$$

**Step 2**: Find $x_{f_{ne}}(x_{h}')$ (i.e. $x_{f_{ne}}$ at $x_{h}'$):

$$x_{f_{ne}}(x_{h}') = \sqrt{(1-k_{c})x_{h}'^{2} + k_{c}x_{\sigma}} = \sqrt{(1-k_{c})(CF_{(shape\_1)}x_{h})^{2} + k_{c}x_{\sigma}}$$

$$x_{f_{ne}}(x_{h}') = \sqrt{0.68855 (0.86603x_{h})^{2} + 0.31145x_{\sigma}}$$

**Step 3**: Find $x_{f_{ne}}' = CF_{(shape\_2)}x_{f_{ne}}(x_{h}') = x_{f_{no}}$

$$x_{f_{no}} = x_{f_{ne}}' = 1.099x_{f_{ne}}(x_{h}')$$

The three steps above are illustrated graphically in Figure 4.10 for the case of $x_{\sigma} = 1$, similarly for other values of $x_{\sigma}$. In Figure 4.10, the 4th graph compares the graph obtained from the three steps above with the one calculated by equation (4.29).

If $x_{\text{mode}} = x_{L} = x_{\sigma} = 1$ ($n' = 1$, $L' = 5$ m, $\sigma' = 5$ MPa):

Figure 4.10 – Constructing the graph of frequency ratio vs. height ratio of circular cross-section from rectangular cross-section based on theoretical equation
4.4.2 Combined Effect of Bending Stiffness and Length

When only the length and the tensile stress of the “original” tie-rod is changed, meaning \( x_{\text{mode}} = x_{h} \) (or \( x_{d} \)) = 1, \( x_{L} \neq 1 \) and \( x_{\sigma} \neq 1 \):

\[
\begin{align*}
\text{Rectangular cross-section} & : \\
 x_{f_{\text{nc}}} & = \frac{1}{x_{L}} \sqrt{(1 - k_{\alpha}) \left( \frac{1}{x_{L}} \right)^{2} + k_{\alpha} x_{\sigma}} \tag{4.31} \\
\Rightarrow x_{f_{\text{nc}}} & = \frac{1}{x_{L}} \sqrt{0.68855 \left( \frac{1}{x_{L}} \right)^{2} + 0.31145 x_{\sigma}} \\
\text{Circular cross-section} & : \\
 x_{f_{\text{no}}} & = \frac{1}{x_{L}} \sqrt{(1 - k_{\alpha}) \left( \frac{1}{x_{L}} \right)^{2} + k_{\alpha} x_{\sigma}} \tag{4.32} \\
\Rightarrow x_{f_{\text{no}}} & = \frac{1}{x_{L}} \sqrt{0.62379 \left( \frac{1}{x_{L}} \right)^{2} + 0.37621 x_{\sigma}}
\end{align*}
\]

Choosing a range of \( L' \) such that \( 1 \text{ m} \leq L' \leq 10 \text{ m} \Rightarrow 0.2 \leq x_{L} \leq 2.0 \). To combine the effect of length and shape of the cross-section, the following steps are taken and illustrated in Figure 4.11:

**Step 1**: Calculate \( x'_{L} = \frac{x_{L}}{\text{CF}_{(\text{shape}_1)}} \)

\( x'_{L} = \frac{x_{L}}{0.86603} \)

**Step 2**: Find \( x_{f_{\text{nc}}}(x'_{L}) \) (i.e. \( x_{f_{\text{nc}}} \) at \( x'_{L} \)):

\[
x_{f_{\text{nc}}}(x'_{L}) = \frac{1}{x'_{L}} \sqrt{(1 - k_{\alpha}) \left( \frac{1}{x'_{L}} \right)^{2} + k_{\alpha} x_{\sigma}}
\]

\[
x_{f_{\text{nc}}}(x'_{L}) = \frac{0.86603}{x_{L}} \sqrt{0.68855 \left( \frac{0.86603}{x_{L}} \right)^{2} + 0.31145 x_{\sigma}}
\]

**Step 3**: Find \( x'_{f_{\text{no}}} = \text{CF}_{(\text{shape}_2)} \frac{1}{\text{CF}_{(\text{shape}_1)}} x_{f_{\text{nc}}}(x'_{L}) = x_{f_{\text{no}}} \)

\[
x_{f_{\text{no}}} = x'_{f_{\text{nc}}} = 1.0991 \frac{1}{0.86603} x_{f_{\text{nc}}}(x'_{L})
\]
If $x_{\text{mode}} = x_h$ (or $x_d$) = $x_{\sigma} = 1$ ($n' = 1$, $h'$ (or $d'$) = 0.04 m, $\sigma' = 5$ MPa):

In conclusion, the effect of different factors can be combined following a similar approach.

4.4.3 Combined Effect of Cross-Section, Length and Mode Number

As shown in the previous Sections, it is possible to find the “correction factors” due to different shapes of cross-section. Similarly, the correction factors for length and order of the mode can be found. Recalling Eq. (4.20) for rectangular tie-rod and similar to the process of finding the correction factors for shape:

$$ x_{fn\text{c}} = \left( \frac{x_{\text{mode}}}{x_L} \right) \sqrt{\left( 1 - k_L \right) \left( \frac{x_{\text{mode}}}{x_L} \right)^2 + k_L \sigma} $$

- Correction factor for length: $CF_{\text{(length)}} = \frac{1}{x_L}$ such that:
  $$\begin{align*}
x'_h(\text{length}) &= CF_{\text{(length)}} x_h \\
x'_{fn\text{c}}(\text{length}) &= CF_{\text{(length)}} x_{fn\text{c}}(x'_h(\text{length}))
  \end{align*}$$

- Correction factor for order of mode: $CF_{\text{(mode)}} = x_{\text{mode}}$ such that:
  $$\begin{align*}
x'_h(\text{mode}) &= CF_{\text{(mode)}} x_h \\
x'_{fn\text{c}}(\text{mode}) &= CF_{\text{(mode)}} x_{fn\text{c}}(x'_h(\text{mode}))
  \end{align*}$$
4.4.4 Summary of the Effect of the Factors Affecting the Frequency of Tie-Rods

A summary of the effects of all factors are presented graphically in the following figures. The vertical axis shows the frequency ratio due to the changes in one or two of the factors. These ratios are for a case study of the “original” tie-rod, whose characteristics are $L = 5$ m, $h = 0.04$ m, $b = 0.04$ m, $\sigma = 5$ MPa, $n = 1$ (i.e. 1st mode) and pinned at two ends.

**Figure 4.12** – Frequency ratio vs. applied tensile stress ratio

**Figure 4.13** – Frequency ratio of $0.04 \times 0.04$ m rectangular tie-rod and $0.02$ m in diameter circular tie-rods at different applied tensile stresses

**Figure 4.14** – Frequency ratio of rectangular and circular tie-rods when $h = d$ vs. height ratio

**Figure 4.15** – Frequency ratio of rectangular tie-rods vs. height ratio
It should be noted that the effect of shape of cross-section refers to the effect when the cross-section is changed from rectangular to circular. In these cases, if the diameter is equal to the height (Figure 4.14), the maximum negative effect is 0.867. In combination with the size effect, the effect of shape of cross-section is more significant when the size is increased and there are lower tensile stresses. If the diameter is not equal to the height, for example when the height is twice the diameter (Figure 4.13), the effect of the shape is very sensitive for low applied tensile stress, in which the maximum negative effect is 0.433.

Regarding the effect of size of cross-section (Figure 4.15), it highly depends on the effect of applied tensile stress. If considering the effect of size of cross-section separately (i.e. $x_\sigma = 1$), the maximum positive effect of size of cross-section is $x_n = x_h$.

When the effect of each factor is combined with the effect of tensile stress, the graphs show that the effects of length and mode order are less dependent on the effect of tensile stress than the shape and size of cross-section. Furthermore, the length and mode order effects are proven to be much more significant, which agree with the results from DIANA.

The ratios as shown in the figures above are compared with the results from DIANA. The results based on the theoretical equations have proved that the results from DIANA are reliable and the estimated maximum effects in Table 4-14 are correct.
Therefore, the results in Table 4-14 are updated in Table 4-16 below with analytical proof, except for the effect of boundary condition.

Table 4-16
Factors affecting the frequency of tie-rods based on theoretical equations

<table>
<thead>
<tr>
<th>No.</th>
<th>Factor</th>
<th>Effect*</th>
<th>Estimated maximum effect</th>
<th>Level of influence*</th>
<th>Level of dependence*</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>Tensile stress</td>
<td>(+)</td>
<td>$\sqrt{\lambda \sigma}$</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>02</td>
<td>Boundary condition</td>
<td>(+)</td>
<td>2.267</td>
<td>Medium</td>
<td>Low</td>
</tr>
<tr>
<td>03</td>
<td>Size of cross-section</td>
<td>(+)</td>
<td>$x_h$</td>
<td>Medium</td>
<td>High</td>
</tr>
<tr>
<td>04</td>
<td>Order of the mode</td>
<td>(+)</td>
<td>$(\times_{\text{mode}})^2$</td>
<td>High</td>
<td>Medium</td>
</tr>
<tr>
<td>05</td>
<td>Length</td>
<td>(-)</td>
<td>$1/\lambda L^2$</td>
<td>High</td>
<td>Medium</td>
</tr>
<tr>
<td>06</td>
<td>Shape of cross-section (when $h = d$)</td>
<td>(-)</td>
<td>0.867</td>
<td>Medium</td>
<td>High</td>
</tr>
</tbody>
</table>

The effect of boundary condition is not assessed analytically because there are no closed form solutions for the theoretical equations of fixed-pinned and fixed-fixed conditions. However, based on the consistent results from 288 case studies in DIANA, the factors in Table 4-6 and Table 4-7 can be used to calculate the frequencies of the other boundary conditions based on the frequency of pinned-pinned end condition.

4.5 Estimation of Tensile Force of Tie-Rod

4.5.1 Estimation of Tensile Force of Tie-Rod using Equation

Eq. (4.1) is used to calculate the frequency of tie-rod in pinned-pinned condition and match closely with the results from DIANA and SAP2000 programs

$$f_n = \frac{n}{2} \sqrt{\frac{n^2 \pi^2 EI}{\bar{m} L^4} + \frac{T}{\bar{m} L^2}},$$

$$\Rightarrow T = \frac{4\bar{m} L^2 f_n^2}{n^2} - \frac{n^2 \pi^2 EI}{L^2}$$

where $T$ is the tensile force (N), $f_n$ is the frequency of mode $n^{th}$ (Hz); $\omega_n$ is the circular frequency of mode $n^{th}$ (rad); $n$ is the mode number; $L$ is the length of the tie-rod (m); $EI$ is the bending stiffness (Nm²) and $\bar{m}$ is mass per unit length of the tie-rod (kg/m).
To take into account the boundary conditions in a conservative approach, choosing the worst factor between the frequencies of the fixed-fixed and pinned-pinned condition, which is 2.267. Approximately, the tensile force $T$ is always in a range when $f_n$ is in the range from 1.0 to 2.267 times the frequency of pinned-pinned condition

$$
T = \frac{4mL^2}{n^2} f_n^2 - \frac{n^2 \pi^2 EI}{L^2} \quad \left\{ \begin{array}{l}
1.0 f_{n \text{ (pinned–pinned)}} \leq f_n \leq 2.267 f_{n \text{ (pinned–pinned)}}
\end{array} \right.
$$

where $f_{n \text{ (pinned–pinned)}}$ is the frequency of mode $n^{th}$ in pinned-pinned condition (Hz).

### 4.5.2 Construction of Standard Charts

The purpose of this Section is to construct a chart or several charts showing the relationship between the frequency and tensile stress of a tie-rod and other variables. The charts should be able to cover a wide range of different tie-rods.

Based on the study of the combination of the effects of different factors in previous sections and the results of the “correction factors”, it is possible to propose a standard chart. The correction factors are used to relate the tie-rods of different characteristics to the ones already known in the chart. The procedures below describe the methodology to estimate the tensile stress of a random tie-rod based on chart I in Figure 4.18.

Assuming a tie-rod, named “tie-rod 1” has the following characteristics: length $L_1$ (m), cross-section with a height of $h_1$ (m) (rectangular) or diameter of $h_1$ (m) (circular), measured frequency of the $n^{th}$ mode $f_1$ (Hz). The boundary condition is assumed to be pinned-pinned. The assumed material properties are $E = 210$ GPa and $\rho_s = 7850$ kg/m$^3$. The task is to find the tensile stress of the tie-rod based on chart I.
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Figure 4.18 – Construction of standard chart I (pinned-pinned end condition)

**Step 1**: Calculate $h_1/L_1$ and $L_1/5$;

**Step 2**: For a rectangular cross-section: if $L_1 = 5$ m and $n_1 = 1$ (1st mode), directly use the chart I with $f_1$ and $h_1/L_1$ to determine the tensile stress – ignore steps 2a, 2b and 2c;

**Step 2a**: For a rectangular cross-section: if $n_1 = 1$ and $L_1 \neq 5$ m, divide $f_1$ by $(L_1/5)$:

$$f_{1\text{ corrected}} = \frac{f_1}{(L_1/5)} = \frac{5}{L_1} f_1 \text{ (Hz).}$$

(Using correction factor for length);

Use chart I with $f_{1\text{ corrected}}$ and $h_1/L_1$ to find the tensile stress – ignore steps 2b and 2c;

**Step 2b**: For a rectangular cross-section: if $n_1 \neq 1$ and $L_1 \neq 5$ m:

$$f_{1\text{ corrected}} = \left(\frac{L_1}{5 n_1}\right) f_1$$

(Using correction factors for length and order of mode);

Use chart I with $f_{1\text{ corrected}}$ and $(h_1/L_1)\text{ corrected}$ to find the tensile stress – ignore step 2c;

**Step 2c**: If the cross-section is circular and $n_1 \neq 1$ and $L_1 \neq 5$ m:

$$f_{1\text{ corrected}} = \left(\frac{1}{CF_{\text{shape, 2}}}\right) \left(\frac{L_1}{5 n_1}\right) f_1 = 0.9098 \left(\frac{L_1}{5 n_1}\right) f_1$$

(Using correction factors for shape, length and order of mode);

Then use chart I with $f_{1\text{ corrected}}$ and $(h_1/L_1)\text{ corrected}$ to find the tensile stress.
4.5.3 Proposed Standard Charts

The chart I described in the Section above is for tie-rods of 5 m length. For simplicity, the length, \( L \) of the proposed standard charts should be changed to 1 m. The proposed standard charts for tie-rods with three boundary conditions are presented in Figure 4.19, Figure 4.20 and Figure 4.21.

The proposed standard chart I-a for pinned-pinned end connection is based on Eq. (4.1) with a set of \( h/L \) values ranging from 0.0004 to 0.04 and other parameters: \( n = 1, L = 1 \text{ m}, \sigma = 0 – 200 \text{ MPa}, E = 210 \text{ GPa}, \rho_s = 7850 \text{ kg/m}^3 \). The charts I-b and I-c for fixed-pinned and fixed-fixed end conditions are calculated by multiplying the frequencies of pinned-pinned condition by the factors of mode 1 (the worse cases) in Table 4-6 and Table 4-7.

![Figure 4.19 – Standard Chart I-a (pinned-pinned condition)](image1)

![Figure 4.20 – Standard Chart I-b (fixed-pinned condition)](image2)
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Correction factors for the values of $h/L$ ratio and frequency $f$ to use the standard charts:

- Rectangular cross-section:
  \[
  \left( \frac{h}{L} \right)_{\text{corrected}} = \left[ n \right] \left( \frac{h_1}{L_1} \right) \\
  f_{n,\text{corrected}} = \left[ \frac{1}{n_1} \right] L_1 f_1
  \]

- Circular cross-section:
  \[
  \left( \frac{h}{L} \right)_{\text{corrected}} = \left[ 0.8660 \left( \frac{n_1}{n_1} \right) \right] \left( \frac{h_1}{L_1} \right) \\
  f_{n,\text{corrected}} = \left[ 0.9098 \left( \frac{1}{n_1} \right) L_1 \right] f_1
  \]

where $h_1$, $L_1$, $n_1$, $f_1$ are the parameters of an arbitrary tie-rod whose tensile stress is to be determined by the standard charts Ia, Ib or Ic; $(h_1/L_1)_{\text{corrected}}$ and $f_{1,\text{corrected}}$ are the values to be used for the standard charts.

### 4.5.4 Verification of the Proposed Standard Charts

An arbitrary tie-rod was chosen to verify the proposed standard charts. A tie-rod with the length of 10 m and 3x4 cm rectangular cross-section and pinned-supported at both ends, its frequencies was obtained using DIANA. The frequency of the 4th mode at 25 MPa tensile stress is 16.02 Hz.

To verify the value of the tensile stress by using the proposed standard chart I-a for pinned-pinned end condition:

- Calculate $h/L = 0.03/10 = 0.003$;
- $(h/L)_{\text{corrected}} = n \times (h/L) = 4 \times (h/L) = 0.012$;

![Figure 4.21 – Standard Chart I-c (fixed-fixed end condition)](image-url)
• \( f_{n,\text{corrected}} = \frac{1}{n} \times L \times f = \frac{1}{4} \times 10 \times f_4 = \frac{1}{4} \times 10 \times 16.02 = 40.06 \text{ Hz.} \)

Figure 4.22 shows the value of tensile stress determined in the Chart I-a. The tensile stress is 25 MPa which is accurate.

![Figure 4.22 – Verification of the proposed standard chart I-a](image)

### 4.6 Conclusion

In this chapter, the modal analysis was performed for tie-rods of different characteristics using FE program (DIANA, 2008). There were a total of 324 case studies conducted. The results from DIANA of the first six modes match well with theoretical approach and SAP2000 program.

Regarding the modes of vibration, the effect of tensile stress is more significant when the boundary condition is fixed-fixed and fixed-pinned compared to pinned-pinned condition, as well as for the lower modes and longer lengths.

There are six factors affecting the frequency of tie-rod. Among them, the effect of length and the order of the mode are the most significant, followed by the effect of the size of the cross-section, boundary condition, tensile stress and shape of the cross-section.

The length and shape of the cross-section have negative effects to the frequency, whereas all other factors have positive effects. In particular, the larger the cross-section, the more restrained the boundary condition and the higher the applied tensile stress, the higher the
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frequency. In addition, the longer the tie-rod, the lower the frequency and the frequency is lower when the cross-section is changed from rectangular to circular cross-section assuming the same size.

The estimated maximum effects are first calculated numerically from DIANA results. After that, they were verified analytically based on theoretical equations for pinned-pinned condition. The results proved to be reliable.

Expressing the change in the factors as ratios, let $x_h$ be the height ratio between the new and original height, similarly for $x_{\text{mode}}$, $x_{\sigma}$, $x_L$, and $x_{fn}$ be the frequency ratio due to the changes in the factors. When considering the effects of the factors separately, the maximum positive effects are such that $x_{fn}$ is equal to $x_{\text{mode}}^2$, $x_h$, 2.267, $\sqrt{x_{\sigma}}$ corresponding to mode order, size of cross-section, boundary condition and tensile stress. The maximum negative effects are $1/x_L^2$ and 0.867 corresponding to the length and shape of cross-section.

Regarding the boundary condition effect, except the pinned-pinned end condition, the results are based on numerical results from DIANA only because no closed-form solution of theoretical equations could be found for the fixed-pinned and fixed-fixed conditions. The frequency ratios between different boundary conditions were calculated taking the pinned-pinned condition as reference. The results from 288 case studies are highly consistent, proving that the boundary condition effect is not very dependent on other factors’ effects. Therefore, the frequency ratios in Table 4-6 and Table 4-7 can be used as multiplying factors with reasonably high accuracy.

To estimate the tensile stress of the tie-rod, the charts I-a, I-b and I-c can be used or the theoretical Eq. (4.34). The construction of the standard charts is based on the combined effect of different factors. There can be standard charts taking into account all the factors’ effects. However, for more accuracy, the effect of the mode order should be separated for fixed-pinned and fixed-fixed condition. As a result, charts I-b and I-c should be used for separate modes and to utilize the multiplying factors for each mode in Table 4-6 and Table 4-7.

It should be noted that the range of the frequency in the theoretical equation is from 1.0 to 2.267 times the frequency of pinned-pinned condition; therefore, a range of the tensile stress is estimated in Eq. (4.34). To estimate the frequency and tensile stress of real tie-rods, the numerical results will be compared with the experimental results in Chapter 5.
Chapter 5
Experimental Model Analysis Tests in Laboratory

Abstract

After performing the numerical analysis, the results must be verified with experimental results of real tie-rods in laboratory or existing structures. This chapter presents the results of the laboratory tests of four different tie-rods. In total, there were 84 tests. For comparison, three different types of sensors: conventional, wireless and laser vibrometer were used. The results of the wireless accelerometer and laser vibrometer system are compared with that of the wired accelerometer. Then, the experimental results will be compared with numerical results to assess the accuracy of the theoretical approach. After that, the results will be calibrated in the next Chapter to accurately estimate the tensile stress in tie-rods. In addition,
5.1 Experimental specimen and set-up diagram

To verify the numerical results, four tie-rods of different characteristics were tested in laboratory. Using modal analysis tests, the frequency and the first five to eight modes of vibration were determined at different tensile stresses.

The characteristics of the four tie-rods are presented in Table 5-1.

Table 5-1
Tie-rod specimens for laboratory tests

<table>
<thead>
<tr>
<th>Specimen no.</th>
<th>Length, L (m)</th>
<th>Cross-section</th>
<th>E (GPa)</th>
<th>ρ (kg/m3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Shape</td>
<td>Size, h, b or d (m)</td>
<td></td>
</tr>
<tr>
<td>Tie-rod S1</td>
<td>2.7</td>
<td>□ Square</td>
<td>0.04 x 0.04</td>
<td>210</td>
</tr>
<tr>
<td>Tie-rod S2</td>
<td>5.4</td>
<td>□ Square</td>
<td>0.04 x 0.04</td>
<td>210</td>
</tr>
<tr>
<td>Tie-rod C3</td>
<td>2.7</td>
<td>○ Circular</td>
<td>Ø0.02</td>
<td>210</td>
</tr>
<tr>
<td>Tie-rod C4</td>
<td>5.4</td>
<td>○ Circular</td>
<td>Ø0.02</td>
<td>210</td>
</tr>
</tbody>
</table>

For each tie-rod, the tests were performed with two boundary conditions to stimulate the boundary conditions of real tie-rods. Figure 5.1 and Figure 5.2 show the set up of these two boundary conditions and details of the fixing nuts that were used. The first type of boundary condition is considered to be close to a pinned-pinned condition (PP). Whereas the second type is closer to a fixed-fixed condition (FF).

Figure 5.1 – Details of female nuts type A and B and end of rectangular tie-rod specimens
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Boundary condition type PP

Boundary condition type FF

Figure 5.2 – Test diagrams of two types of boundary conditions: PP and FF
5.2 Experimental introduction and equipment

The dynamic tests performed are output-only experimental modal identification tests. They are also called Operational Modal Analysis (OMA). The tests consist of fixing the sensors (accelerometers) and recording the response of the structure to random impacts using the Data Acquisition (DAQ) equipment.

The accelerometer is able to transform a physical quantity that defines the system response in terms of accelerations. Conventional wired accelerometers were used in all tests. Six wired accelerometers were used for 5.4 m-long tie-rods, and four were used for 2.7 m-long tie-rods. Figure 5.3 shows the arranged positions of the wired accelerometers on tie-rod specimens. It is important to position the accelerometers to avoid the locations of zero displacement for each mode of vibration of tie-rods. Figure 5.3 shows the first six modes of vibration and the locations of the accelerometers.

![Diagram showing positions of accelerometers on tie-rods of 5.4 m and 2.7 m]

Figure 5.3 – Determination of the positions of wired accelerometers
In some tests, wireless accelerometers and laser vibrometer equipment were also used to compare their results with those of the wired accelerometers and to examine the working feasibility of the wireless and laser systems. The two types of accelerometer systems (wired and wireless) and the vibrometer system are presented in Figure 5.4.

Figure 5.4 – Three types of systems (a) Wired accelerometer system, (b) Wireless accelerometer system, (c) Laser vibrometer system
As seen in Figure 5.4, compared with the wired accelerometer system, the wireless accelerometer system has the advantages of smaller size and the absence of cables connecting the WP DAQ to a computer. These features provide great convenience and are highly applicable for historical monuments and all other structures. The gateway mote can be placed at a distance as far as 50 m from the testing location. For the wireless system, three accelerometers were used.

For the laser vibrometer system, the equipment is Polytech Fiber-Optic Laser Vibrometer OFV-5000. It analyzes the vibrational behavior of the tie-rod using two sensor heads jointed with each other. The advantages are the system is non-destructive and non-contact, which are highly applicable to historical structures as well. The sensor can absorb light at a distance as far as 30 m. The results of all the systems will be discussed Sections 5.5.3 and 5.5.4.

Figure 5.5 shows the preparation procedure to set up the laser vibrometer system. It requires focusing the sensor heads to acquire full signals on the vibrometer controller. The position of the reflected point of the sensor head’s lighting was adjusted to coincide with that of one wired accelerometer in order to compare the results of the two systems.

![Figure 5.5 – Setting-up the laser vibrometer system: (a) Focusing the sensor heads; (b) The system with full signals](image)
Table 5-2 tabulates the instrumentation and test settings used for the modal testing. For each test, two or three frequency response functions (FRFs) were measured. The tie-rod was excited by applying a vertical impact by hand. The impacts were applied every 5 seconds at different points along the tie-rod and an averaged FRF is computed for each impact point. A standard modal analysis was performed on the collected data set of transfer functions and modal parameters for the structure are extracted.

<table>
<thead>
<tr>
<th>System</th>
<th>Specimen</th>
<th>Parameter</th>
<th>Setting</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wired accelerometer</td>
<td>2.7 m-long tie-rods</td>
<td>Sample frequency</td>
<td>800</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>60</td>
<td>seconds</td>
</tr>
<tr>
<td></td>
<td>5.4 m-long tie-rods</td>
<td>Sample frequency</td>
<td>500</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>60</td>
<td>seconds</td>
</tr>
<tr>
<td>Fiber laser vibrometer</td>
<td>2.7 m-long tie-rods</td>
<td>Sample frequency</td>
<td>800</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>60</td>
<td>seconds</td>
</tr>
<tr>
<td></td>
<td>5.4 m-long tie-rods</td>
<td>Sample frequency</td>
<td>500</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>60</td>
<td>seconds</td>
</tr>
<tr>
<td>Wireless accelerometer</td>
<td>2.7 m-long tie-rods</td>
<td>Sample frequency</td>
<td>100</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>30</td>
<td>seconds</td>
</tr>
<tr>
<td></td>
<td>5.4 m-long tie-rods</td>
<td>Sample frequency</td>
<td>100</td>
<td>Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sample length</td>
<td>30</td>
<td>seconds</td>
</tr>
</tbody>
</table>

For each specimen, the tests were repeated with two boundary conditions, as mentioned in Figure 5.2, and four or five values of tensile stress depending on the specimens. The tensile forces were applied manually using a wrench to tighten the specimens at one end. To control the values of the applied tensile forces or tensile stresses in the tie-rod, the tests were performed by displacement-control using two LVDTs. From the measured displacements of the LVDTs, the strains were calculated, and so were the tensile stresses.

The values of the LVDTs’ displacements were recorded every 1/3 second. The LVDTs were placed on both side of a tie-rod at the same location to take the average value of the two measurements. They were placed parallel to and at the same level as the center line of the tie-rod’s cross-section to avoid the effect of bending moment due to self-weight. In most of the tests, the LVDTs were placed near the right end. The LVDT system is shown in Figure 5.6.
Other equipments used in the tests include a hand level on the rectangular tie-rods and the laser meter to measure the deflections at several points along the tie-rods with reference to the floor of the supporting frame (Figure 5.7).

5.3 Experimental set-up and performance

5.3.1 Overview of the tests

Figure 5.8 and Figure 5.9 show tie-rod S1 which was tested with two boundary conditions along with the supporting frame structure. The 0.04x0.04 m square rod with the length of 2.7 m was first set up to be close to pin-connected to the frame (boundary condition type PP) and subjected to the change in tensile stresses of approximately 0 MPa, 20 MPa, 40 MPa and 60 MPa with reference to its initial stress condition. Then the tests were repeated with the boundary condition type FF in which the tie-rod was more fixed-connected to the frame. In total, there were 9 tests conducted for tie-rod S1.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure 5.8 – Tests of tie-rod S1 with boundary condition (BC) type PP

Figure 5.9 – Tests of tie-rod S1 with boundary condition (BC) type FF

Figure 5.10 shows the test S2-PP II with the wireless system, which was the only test when the LVDTs were placed in the middle. An example of the tests with the laser vibrometer system is shown in Figure 5.11. The set-up of all other tests is given in Appendix A.1.
Figure 5.10 – Tests of tie-rod S2 with boundary condition (BC) type PP and wireless accelerometer system and LVDTs in the middle of tie-rod

Figure 5.11 – Tests of tie-rod C4 with boundary condition (BC) type PP with laser vibrometer system
5.3.2 Boundary condition

5.3.2.1 Square tie-rods S1 and S2

Figure 5.12 and Figure 5.13 present in details the two types of boundary conditions of square tie-rods. In the boundary condition type PP, by applying an initial tensile stress to the tie-rod, the tie-rod was placed above and not sitting on the holes of the supporting frames. This arrangement was maintained throughout the tests. In type FF, a steel hollow cylinder was inserted inside the holes of the supporting frames.

Figure 5.12 – Boundary condition type PP of Tests S1-PP and S2-PP: (a), (b) View of the right end

Figure 5.13 – Boundary condition type FF of Tests S1-FF: (a), (b) View of the right end; (c) View of the left end; (d) Applying tensile forces at the right end of square tie-rod
5.3.2.2 Circular tie-rods C3 and C4

Figure 5.14 and Figure 5.15 present the two types of boundary conditions of circular tie-rods. They are similar to those of square tie-rods. However, the tensile forces were applied at the left end of the tie-rod to avoid the rotating influence on the LVDT system placed at the right end. It is easier to accidentally rotate the circular tie-rods during stressing than the square specimens.

Figure 5.14 – Boundary condition type PP of Tests C3-PP and C4-PP: (a), (b) View of the right end

Figure 5.15 – Boundary condition type FF of Tests C3-FF and C4-FF: (a) View of the right end; (b), (c) View of the left end; (d) Applying the tensile forces at the left end of circular tie-rod
5.3.3 The LVDT system

5.3.3.1 Square tie-rods S1 and S2

The two LVDT systems are shown in Figure 5.16. The anchor plates of the LVDTs were glued to the tie-rod. Before applying the glue, the surface of the tie-rod was cleaned to remove any paint coating. Furthermore, the glue should be applied only as a thin line at the center-line of the tie-rod’s cross-section, so that the effect of the glue on the values of the LVDTs will be reduced to a minimum. To calculate the displacement, $\Delta L$, when the strain, $\varepsilon$, is known, the original length $L_0$ was measured from the middle of the anchor plates. The $L_0$ is equal to 150 mm.

![Figure 5.16](image)

Figure 5.16 – LVDTs system of Tests S1-PP, S1-FF, S2-PP and S2-FF: (a) Front view of the first LVDT; (b) Top view of the two LVDTs

5.3.3.2 Circular tie-rods C3 and C4

In all tests, the two LVDTs were placed near the right end of the tie-rod and the tensile forces were applied at the left end. To fix the positions and directions of the LVDTs on the circular tie-rods, two anchor plates were used for each LVDT as shown in Figure 5.17.

![Figure 5.17](image)

Figure 5.17 – LVDTs system of Tests C3-PP, C3-FF, C4-PP and C4-FF: (a) Front view of the first LVDT; (b) Top view of the two LVDTs
5.3.4 Controlling the value of tensile stress

A procedure for controlling the tensile forces applied to the tie-rod specimens is

- Before applying the tensile force, recording the measured values of two LVDTs, $a_1$ and $b_1$;
- Calculating the desired tensile stress, $\sigma_1$;
- Calculating the corresponding strain, $\varepsilon_1 = \sigma_1 / \varepsilon_1$, then the corresponding change in displacement, $\Delta L_1 = \varepsilon_1 L_0$;
- Calculating the desired values of two LVDTs by plusing or minusing the measured values of the LVDTs with the desired change in displacement, $a_2 = a_1 + \Delta L_1$ and $b_2 = b_1 - \Delta L_1$ ("+" when applying and "-" when releasing the applied tensile force);
- Applying the tensile force and observing the measured values of the LVDTs at the same time, using the desired values $a_2$ and $b_2$ as reference;
- During the process, checking the current tensile stress (average value) based on the updated values of the LVDTs;
- Stopping the application of the tensile force once the average value of the calculated tensile stress is equal to the desired one (i.e. $\sigma_1$).

5.3.5 Experimental procedure

A general procedure for the test is: first, the specimen was measured to double-check the length and size of the cross-section; then the specimen was mounted to the supporting frame with the target boundary condition. The tie-rod in this condition was subjected to its own self-weight only without any applied tensile force. After that, the wired accelerometers, the LVDTs or other systems were set up. The test was started by lifting up the specimen using a crane in the middle (Figure 5.18). The values of the LVDTs were recorded. An initial tensile force was applied manually to both ends, in the case of boundary condition type PP, to hold the bars above the holes of the supporting frame, so that the tie-rod was supported by the female nuts type A. In both cases of the two boundary conditions, this procedure partly eliminated the effect of bending curvature due to self-weight. Then the crane was removed, and the updated values of the LVDTs were recorded. The difference between the values indicates the initial strain and the corresponding initial tensile stress within the tie-rod due to the applied force.

After that, the modal analysis test was performed with a desired value of the change in the tensile stress. Let $\Delta_0$ be the change in the tensile stress with reference to the tie-rod’s initial stress condition $\sigma_0$. The first test was run for $\Delta_0 = 0$ MPa. By applying a vertical impact by
hand every 5 seconds at different points along the tie-rod. The time-dependent accelerations obtained by the six wired accelerometers were recorded. The results were processed and the Frequency Response Spectrum (FRS) was obtained. The peaks of the FRSs correspond to the frequency values of the tie-rod. The test was repeated with the same $\Delta \sigma_0$ so that the average values of two or in some cases, three FRFs can be used. The deflections of the tie-rod at $\sigma_0$ and for each increment of $\Delta \sigma$ were measured and recorded.

Consequently, the same procedures repeated by increasing $\Delta \sigma$. The tensile forces were applied manually at one end of the tie-rod. After applying the tensile force, the hand level was used to verify the rectangular tie-rod was level to avoid any rotation of the cross-section.

Figure 5.19 shows an example of four different stages of loads that were applied to the tie-rod S1. The first stage corresponds to the tie-rod in its initial tension stress condition, the next three stages correspond to the application of successive load increments, $\Delta \sigma$.

Figure 5.19 – Stages of loads in the tests performed on the tie rod S1
5.3.6 Summary of all the tests

Table 5-3 summarizes the tests that were performed with the position of the LVDT system, the location of the applied tensile force and the stages of load in terms of $\Delta \sigma$. Also the use of different sensor systems is indicated. There were a total of 84 tests conducted.

Table 5-3
Summary of the laboratory tests performed

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Location of LVDTs</th>
<th>Position of applied $\sigma$</th>
<th>$\Delta \sigma$ (MPa) (approximately)</th>
<th>System</th>
<th>Wired acceler.</th>
<th>Wireless acceler.</th>
<th>Laser vibro.</th>
<th>No. of FRFs obtained</th>
<th>Total no. of tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PP</td>
<td>Near the right end</td>
<td>The right end</td>
<td>0, 20, 40 and 60</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 or 3 per $\Delta \sigma$</td>
<td>9</td>
</tr>
<tr>
<td>S1-FF</td>
<td>Near the right end</td>
<td>The right end</td>
<td>0, 20, 40 and 60</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 per $\Delta \sigma$</td>
<td>8</td>
</tr>
<tr>
<td>S2-PP</td>
<td>Near the right end</td>
<td>The right end</td>
<td>0, 20, 40 and 60</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 per $\Delta \sigma$</td>
<td>8</td>
</tr>
<tr>
<td>S2-PP_II</td>
<td>In the middle</td>
<td>The right end</td>
<td>0, 20, 40 and 60</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td></td>
<td>2 per $\Delta \sigma$</td>
<td>8</td>
</tr>
<tr>
<td>S2-FF</td>
<td>Near the right end</td>
<td>The right end</td>
<td>0, 20, 40 and 60</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td></td>
<td>2 per $\Delta \sigma$</td>
<td>8</td>
</tr>
<tr>
<td>C3-PP</td>
<td>Near the right end</td>
<td>The left end</td>
<td>0, 20, 40, 60 and 80</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 per $\Delta \sigma$</td>
<td>10</td>
</tr>
<tr>
<td>C3-FF</td>
<td>Near the right end</td>
<td>The left end</td>
<td>0, 20, 40, 60 and 80</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 or 3 per $\Delta \sigma$</td>
<td>11</td>
</tr>
<tr>
<td>C4-PP</td>
<td>Near the right end</td>
<td>The left end</td>
<td>0, 20, 40, 60 and 80</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td></td>
<td>2 or 3 per $\Delta \sigma$</td>
<td>11</td>
</tr>
<tr>
<td>C4-FF</td>
<td>Near the right end</td>
<td>The left end</td>
<td>0, 20, 40, 60 and 80</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
<td>2 or 3 per $\Delta \sigma$</td>
<td>11</td>
</tr>
</tbody>
</table>

where $\Delta \sigma$: change in the value of the tensile stress (MPa);

Test no.: “S1 – PP a”

Specimen

Test corresponds to a value of $\Delta \sigma$

Boundary condition
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Table 5-4 summarized the values of the initial stress due to the applied tensile force, $\sigma_0$ (MPa) and the exact changes in the tensile stress with reference to $\sigma_0$, $\Delta_\sigma$ (MPa) of all the tests.

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Specimen</th>
<th>Initial tensile stress, $\sigma_0$ (MPa)</th>
<th>Change in the tensile stress, $\Delta_\sigma$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PP</td>
<td>2.7 m</td>
<td>5.25</td>
<td>19.6</td>
</tr>
<tr>
<td>S1-FF</td>
<td>2.7 m</td>
<td>2.54</td>
<td>20.3</td>
</tr>
<tr>
<td>S2-PP</td>
<td>5.4 m</td>
<td>8.4</td>
<td>21</td>
</tr>
<tr>
<td>S2-FF</td>
<td>5.4 m</td>
<td>4.2</td>
<td>20.3</td>
</tr>
<tr>
<td>C3-PP</td>
<td>2.7 m</td>
<td>5.6</td>
<td>20.3</td>
</tr>
<tr>
<td>C3-FF</td>
<td>2.7 m</td>
<td>0</td>
<td>20.3</td>
</tr>
<tr>
<td>C4-PP</td>
<td>5.4 m</td>
<td>7.7</td>
<td>22.4</td>
</tr>
<tr>
<td>C4-FF</td>
<td>5.4 m</td>
<td>0</td>
<td>20.3</td>
</tr>
</tbody>
</table>

5.4 Discussion about the Effect of Experimental Uncertainty

The laboratory testing data of the four tie-rod specimens will be used to verify the practicality of the theoretical formula or numerical models to estimate the tensile force in tie-rods. In order to use the experimental results as the right reference set, the uncertainties during the test performance should be taken into account. According to Park et al. (2006), it is expected in field applications that there would be some deviations due to uncertainties in measurement data or structural properties. This section aims to validate the accuracy of the experimental results; therefore, several types of uncertainty will be discussed below.

The first type of uncertainty was associated with the magnitude of applied tensile forces creating tensile stresses in the tie-rod specimens. The tensile forces were applied manually using a skew wrench and controlled based on the average measured displacements of two LVDTs. As the values of the changes in the displacements were small (i.e. 0.0143 mm corresponding to an increase of 20 MPa in tensile stress), the measured values of the LVDTs taken were accurate to the third fractional number. In addition, the displacements might be affected by other sources of vibration or noise, which could be neglected in measurements with higher values of the changes in displacements.

The second type was associated with the non-uniform distribution of the applied tensile force on the tie-rod’s cross-section. As a result, the strains measured on both sides of the tie-rod at the same location were not equal, which ideally should be equal to each other.
The third type was related to the possible deviation of the LVDTs from the desired positions parallel to and at the same level as the center line of the tie-rod’s cross-section. This could result from a bending moment due to self-weight of tie-rods on the accuracy of the measured strains.

Another type was the uncertainty associated with the identified natural frequencies. This type of uncertainty may arise when a true natural frequency locates between two obtained frequency responses. On average, two or three frequency responses were acquired for one value of tensile stress.

The fifth type was the possible uncertainty in the set-up of testing accelerometers on top of the circular tie-rods in a same vertical plane, especially they might be twisted when being stressed.

Some other types may include the uncertainty in the material properties of the tie-rod due to production process, which could result in the presence of residual stress in steel; or errors in the damping parameters when applying a vertical impact manually to excite the specimens. However, Park et al. (2006) stated that the effect of the damping change to the resonant frequencies is negligibly small in general.

In conclusion, the first three types of uncertainty have more significant effects on the results than the others because they involved the measurement of small strains, and as a result, affecting the values of the corresponding tensile stresses. The experimental and numerical results will be compared and further conclusions will be made regarding the effect of the uncertainties. The adopted approach will be the results should be combined to assess the estimation of tensile force of real tie-rods.

5.5 Experimental results

5.5.1 Obtaining the Frequency Response Spectrum

Figure 5.20 shows examples of the results of acceleration versus time for tests S2-FF at two values of tensile stress: 4.2 MPa and 44.1 MPa. By performing the Discrete Fourier Transform (DFT) analysis using the Welch method (Welch 1967), the results are transformed from the time domain to the frequency domain, which are called FRS. The results of the FRSs are shown in Figure 5.21, taking the average results of the six wired accelerometers. The frequency values can be obtained by picking up the peaks of the FRSs.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure 5.20 – Acceleration versus time diagram for tests S2-FF

Figure 5.21 – Frequency Response Spectrums (FRSs) of the tests S2-FF
After the excitation by a vertical impact, the tie-rod vibrates with the highest amplitude at the beginning and the amplitude decreases gradually until reaching nearly zero oscillation as seen in Figure 5.20. The signal is like a homogenous damped function.

From the graphs in Figure 5.21, the FRSs for the tie-rod have clear peaks, so that the frequency values can be determined without difficulty. Some side noises could be observed, which were recognized by comparing the FRS obtained from the signals of each accelerometer. The peaks due to the noise should not be taken into account in finding the natural frequencies.

To illustrate the effect of tensile stress on the frequency value, Figure 5.22 presents the overlaying of the FRSs at two values of tensile stress. The FRS is shifted to the right, indicating that the frequencies are increased at higher tensile stress.

![Figure 5.22 – Effect of tensile stress - Frequency Response Spectrums (FRSs) for tests S2-FF](image)

**5.5.2 Values of the tensile stress**

The values of the tensile stress in the experiments will be used for the numerical models to make the comparison between the results. The tensile stresses were calculated based on the displacements measured by two LVDTs. Table 5-5 shows one example of the measured values of the two LVDTs and the corresponding tensile stresses in the tests S2-PP. The values of two LVDTs are not equal. This could be due to the experimental uncertainties as discussed in the previous Section. The average values of the two tensile stresses are used.
5.5.3 Comparison of the results between the wired accelerometer and laser vibrometer system

Figure 5.23 shows the FRSs obtained from the wired accelerometer and laser vibrometer in the tests S2-FF at $\sigma_2 = 44.1$ MPa. The position of the laser vibrometer coincided with the wired accelerometer number 4.

![Wired accelerometer no. 4 and Laser vibrometer](image)

Figure 5.23 – FRSs of the wired accelerometer and laser vibrometer system in the tests S2-FF

It can be seen in Figure 5.23 that the FRS obtained from only one wired accelerometer or laser sensor has more noise and therefore, the peaks are not absolutely clear. The results are improved with more sensors at different points along the tie-rod. The two FRSs are overlain in Figure 5.29.

---

### Table 5-5
Summary of values of the LVDTs and the corresponding tensile stresses in tests S2-PP

<table>
<thead>
<tr>
<th>LVDT 1</th>
<th>LVDT 2</th>
<th>$\Delta L_1$ (mm)</th>
<th>$\Delta L_2$ (mm)</th>
<th>$\sigma_1$ (MPa)</th>
<th>$\sigma_2$ (MPa)</th>
<th>$\sigma_{\text{average}}$ (MPa)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.981</td>
<td>-2.05</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Start of the test</td>
</tr>
<tr>
<td>-1.983</td>
<td>-2.042</td>
<td>-0.002</td>
<td>0.008</td>
<td>-2.8</td>
<td>11.2</td>
<td>4.2</td>
<td>Initial tensile stress, $\sigma_0$</td>
</tr>
<tr>
<td>-1.983</td>
<td>-2.042</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Reference values to start measure frequency, $f$</td>
</tr>
<tr>
<td>-1.969</td>
<td>-2.025</td>
<td>0.012</td>
<td>0.017</td>
<td>16.8</td>
<td>23.8</td>
<td>20.3</td>
<td>Stress increased, $\Delta \sigma \approx 20$ MPa</td>
</tr>
<tr>
<td>-1.952</td>
<td>-2.014</td>
<td>0.029</td>
<td>0.028</td>
<td>40.6</td>
<td>39.2</td>
<td>39.9</td>
<td>Stress increased, $\Delta \sigma \approx 40$ MPa</td>
</tr>
<tr>
<td>-1.939</td>
<td>-1.997</td>
<td>0.042</td>
<td>0.045</td>
<td>58.8</td>
<td>63</td>
<td>60.9</td>
<td>Stress increased, $\Delta \sigma \approx 60$ MPa</td>
</tr>
</tbody>
</table>

---

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The results of the two systems are very close, there are only some differences in the frequency values of the 8th and 9th modes. The detailed results are given in Table 5-6.

### Table 5-6
Frequency differences between the wired accelerometer and laser vibrometer system in tests S2-FF

<table>
<thead>
<tr>
<th>$\sigma$ (MPa)</th>
<th>1$^{st}$ mode</th>
<th>2$^{nd}$ mode</th>
<th>3$^{rd}$ mode</th>
<th>4$^{th}$ mode</th>
<th>5$^{th}$ mode</th>
<th>6$^{th}$ mode</th>
<th>7$^{th}$ mode</th>
<th>8$^{th}$ mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1.1</td>
<td>-0.7</td>
<td>0.5</td>
<td>-1.4</td>
<td>0</td>
</tr>
<tr>
<td>24.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.7</td>
<td>1.0</td>
<td>0.4</td>
<td>-0.3</td>
</tr>
<tr>
<td>44.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>-1.6</td>
</tr>
<tr>
<td>65.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
<td>-1.0</td>
<td>-3.2</td>
</tr>
</tbody>
</table>

It can be seen that the laser vibrometer sensor provides accurate results compared with the conventional wired accelerometer. The maximum difference is 3.2 %.

### 5.5.4 Comparison of the results between the wired accelerometer and wireless accelerometer system

The results of the wired and wireless accelerometer are compared in Table 5-7. The wireless accelerometer system offers some remarkable advantages as mentioned in Section 5.2. However, it has some disadvantages such as it can only acquire a low frequency range and requires a slightly more time-consuming testing procedure.
The wireless accelerometer system also provides highly accurate results compared with the wired accelerometer system. The maximum difference is 1.1%.

### 5.5.5 Effect of bending curvature due to self-weight

A remark on the experimental results is the effect of bending curvature due to self-weight. In the theoretical study of the dynamic response of tie-rods in Chapter 3, the tie-rod was assumed as a beam with rigid bending stiffness. This means the theoretical analysis was based on a straight tie-rod without considering the deflected shape of the tie-rod due to its self-weight. However, when the modal analysis tests were carried out, it was observed that the tie-rod was highly affected by the bending curvature due to self-weight. Therefore, it is important to assess the effect of bending curvature due to self-weight on the frequency of the tie-rod.

Figure 5.25 illustrates this effect of bending curvature due to self-weight on the 0.02 m diameter circular tie-rod 3 with the length of 5.4 m without any applied tensile force and before setting up the boundary condition.
In the following Sections, the results of the experiments will be first compared with the numerical results without the effect of bending curvature. After that, the results will be compared taking into account the effect of bending curvature. Then the effect of bending curvature due to self-weight on the frequency of tie-rod will be concluded.

### 5.5.6 Comparison of the experimental results with the numerical results excluding the effect of bending curvature due to self-weight

To make a comparison between the experimental and numerical results, the four tie-rods in the laboratory tests were modeled using the program SAP2000. The geometrical characteristics and tensile stresses are the same as the tie-rods in the laboratory tests. The tensile stress, $\sigma$ (MPa) in the numerical models is equal to the initial tensile stress, $\sigma_0$ (MPa) plus the change in the tensile stress, $\Delta \sigma$ (MPa).

The numerical models include the mass of the accelerometers applied at their corresponding locations. The mass of each accelerometer plus the cable is 0.3 kg.

#### 5.5.6.1 Results of tie-rod S1

a. The experimental boundary condition type PP

Table 5-8 compares the frequency values obtained from the experiments with the numerical models. The experimental results are the average values of the results from two or three FRFs.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\sigma_0$ (MPa)</th>
<th>$\sigma = \sigma_0 + \Delta \sigma$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.45</td>
<td>13.53</td>
</tr>
<tr>
<td>2</td>
<td>49.05</td>
<td>51.53</td>
</tr>
<tr>
<td>3</td>
<td>105.63</td>
<td>114.49</td>
</tr>
<tr>
<td>4</td>
<td>180.67</td>
<td>198.41</td>
</tr>
<tr>
<td>5</td>
<td>272.47</td>
<td>307.89</td>
</tr>
</tbody>
</table>

where: “Exp. (PP)”: experimental results with boundary condition type PP;

“Num. (PP)”: numerical results with pinned-pinned end condition.
The results are presented in Figure 5.26 including the numerical results with fixed-fixed condition.

![Graphs showing comparison of experimental and numerical results](image)

**Figure 5.26 – Comparison of the experimental and numerical results for tests S1-PP**

The results of the experiments are reasonably close to but lower than the numerical results with pinned-pinned condition. This could be due to the experimental uncertainty in the measured tensile stress. Also the results are closer at lower modes than higher ones.
The comparison shows that the results of the numerical models with pinned-pinned condition can be used to estimate the tensile stress in tie-rod S1 with maximum error of 12% for the first four modes and 16.8% for the 5th mode.

b. The experimental boundary condition type FF

The experimental results are presented in Figure 5.27 together with the numerical results with pinned-pinned and fixed-fixed conditions.

Figure 5.27 – Comparison of the experimental and numerical results for tests S1-FF
The experimental results with boundary condition FF are compared with the numerical results with pinned-pinned condition in Table 5-9.

Table 5-9
Comparison of the frequency values for tests S1-FF with numerical results

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \sigma_0 ) (MPa)</th>
<th>( f ) (Hz)</th>
<th>( \sigma = \sigma_0 + \Delta \sigma ) (MPa)</th>
<th>Diff. (%)</th>
<th>Diff. (%)</th>
<th>Diff. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.28</td>
<td>13.10</td>
<td>1.3</td>
<td>15.13</td>
<td>16.05</td>
<td>6.1</td>
</tr>
<tr>
<td>2</td>
<td>49.34</td>
<td>51.08</td>
<td>3.5</td>
<td>51.16</td>
<td>54.35</td>
<td>6.2</td>
</tr>
<tr>
<td>3</td>
<td>111.65</td>
<td>114.04</td>
<td>2.1</td>
<td>105.50</td>
<td>117.36</td>
<td>11.2</td>
</tr>
<tr>
<td>4</td>
<td>189.45</td>
<td>197.97</td>
<td>4.5</td>
<td>184.80</td>
<td>201.23</td>
<td>8.9</td>
</tr>
<tr>
<td>5</td>
<td>289.90</td>
<td>307.46</td>
<td>6.1</td>
<td>274.60</td>
<td>310.64</td>
<td>13.1</td>
</tr>
</tbody>
</table>

where: “Exp. (FF)”: experimental results with boundary condition type FF.

The results of the experiments with the boundary condition FF are also close to the numerical results with pinned-pinned condition. This indicates that the difference between the two boundary conditions in the experiments is not significant for tie-rod S1. The possible reasons are the tie-rod has a small cross-section-to-length ratio; hence the effect of bending moment is not significant. Therefore, the effect of the types of boundary conditions is not significant either for this tie-rod. Another reason is the set-up of the boundary conditions in the tests. The boundary condition type FF still allows some rotation in the tie-rod. In addition, the female nut type B, as the first pinned support, was placed at a distance from the end of the tie-rod.

Considering the boundary condition of a real tie-rod clamped inside the masonry wall to be close to one of the two experimental boundary conditions, the results of the numerical models with pinned-pinned condition can be used to estimate the tensile stress in tie-rod S1 with reasonable accuracy.

5.5.6.2 Results of tie-rod S2

a. The experimental boundary condition type PP

Figure 5.28 presents the experimental results for tests S2-PP and the numerical results with pinned-pinned and fixed-fixed conditions. The detailed results in tabular form are given in Appendix A.2.
Similarly to tie-rod S1, the experimental results are also close to the numerical results with pinned-pinned condition. The results are consistent except for the first mode at initial tensile stress. The error for the first mode is 21.6%, whereas in all other cases the maximum error is 11%. The reason is likely associated with the effect of bending curvature due to the tie-rod’s self-weight. It can be seen that this effect is significant on the first mode only at low tensile stress.

Figure 5.28 – Comparison of the experimental and numerical results for tests S2-PP excluding the effect of bending curvature due to self-weight
b. The experimental boundary condition type FF

Figure 5.28 presents the experimental results for tests S2-PP and the numerical results with pinned-pinned and fixed-fixed conditions.

![Graph comparing experimental and numerical results with boundary condition FF](image)

Figure 5.29 – Comparison of the experimental and numerical results of the tests S2-FF excluding the effect of bending curvature due to self-weight

The results of tie-rod S2 with boundary condition type FF agree well with both the numerical results and the theoretical equation or a tie-rod with pinned-pinned condition. Furthermore, the effect of bending curvature due to self-weight can be seen again for the first mode at low tensile stress.
5.5.6.3 Results of tie-rod C3

a. The experimental boundary condition type PP

The experimental results for tests C3-PP and the numerical results with pinned-pinned and fixed-fixed conditions are presented in Figure 5.30. The detailed results are provided in Appendix A.2.

![Comparison of the experimental and numerical results for tests C3-PP excluding the effect of bending curvature due to self-weight](image)

Figure 5.30 – Comparison of the experimental and numerical results for tests C3-PP excluding the effect of bending curvature due to self-weight
The results for circular tie-rod C3 are not well-matched with the numerical results. Further conclusion will be made considering the results of all the tests of circular tie-rods.

b. The experimental boundary condition type FF

Figure 5.31 shows the experimental results for tests C3-FF and the numerical results with pinned-pinned and fixed-fixed conditions. The detailed results are provided in Appendix A.2.
As seen in Figure 5.31, the experimental results with the boundary condition type FF are closer to the numerical results with pinned-pinned condition than the type PP. In both tests, the experimental results of the 4th mode are not consistent with the results of other modes. This could be due to some errors in the experimental results of the 4th mode.

5.5.6.4 Results of tie-rod C4

a. The experimental boundary condition type PP

Figure 5.32 shows the experimental results for tests C4-PP and the numerical results with pinned-pinned and fixed-fixed conditions. The detailed results are provided in Appendix A.2.

![Graphs showing experimental and numerical results for C4-PP tests](image)
Among the four tie-rods, the effect of bending curvature due to self-weight on the first mode of tie-rod C4 at initial tensile stress is the most significant.

b. The experimental boundary condition type FF

The experimental results for tests C4-FF and the numerical results with pinned-pinned and fixed-fixed conditions are shown in Figure 5.33. The detailed results are provided in Appendix A.2.

![Comparison of the experimental and numerical results for tests C4-FF excluding the effect of bending curvature due to self-weight](image-url)
From the results of all the tests, the results of the rectangular tie-rod are more consistent and reasonably close to the numerical results with pinned-pinned condition. The results of circular tie-rod are less agreeable and more complicated; although for the circular tie-rod with shorter length (i.e. tie-rod C3), the results are more consistent with the numerical results when the tensile stress is increased. Moreover, the effect of bending curvature is significant on the first mode of 5.4 m-long tie-rods at low tensile stress. The effect is also more significant for circular tie-rods due to higher slenderness or higher length-to-cross-section ratio.

5.5.7 Effect of the bending curvature due to self-weight at different tensile stress

5.5.7.1 Measured deflected shape of tie-rod S1 at different tensile stress

Figure 5.34 shows the measured deflected shapes of tie-rod S1 at different tensile stresses in the tests with the two boundary conditions PP and FF. The higher tensile stress reduces the deflections. The maximum deflections are 2 mm in both cases. Compared with other tie-rods, tie-rod S1 was the least affected by bending curvature due to its self-weight.

![Figure 5.34 – Measured deflected shapes of tie-rod S1 for tests S1-PP and S1-FF](image)
### 5.5.7.2 Measured deflected shape of tie-rod S2 at different tensile stress

Figure 5.35 presents the measured deflected shapes of tie-rod S2 at different tensile stresses. The maximum deflections are 27 mm and 29 mm corresponding to the boundary condition PP and FF, respectively. The effect of bending curvature due to self-weight is significant for both cases. Except when $\Delta \sigma = 0$ MPa, the values of the deflections of the boundary condition FF are higher than that of the boundary condition PP. When $\Delta \sigma = 0$ MPa, tie-rod S2 was less stressed initially in the boundary condition FF than PP, resulting in a smaller deflection. To conclude, the values of the initial tensile stress should be taken into account when comparing the two boundary conditions.

Figure 5.35 also illustrates the effect of the types of boundary conditions, proving that in the boundary condition FF, the tie-rod is more sensitive to the changes in the tensile stress than in the boundary condition PP.

![Figure 5.35 – Measured deflected shapes of tie-rod S2 for tests S2-PP and S2-FF](image)

### 5.5.7.3 Measured deflected shape of tie-rod C3 at different tensile stress

Figure 5.36 shows the measured deflected shapes of tie-rod C3 at different tensile stresses. The maximum deflections are 6.8 mm and 3 mm, respectively. Although the tie-rod was affected by bending curvature due to its self-weight, the effect was not significant compared to tie-rod S2.
5.5.7.4 Measured deflected shape of tie-rod C4 at different tensile stress

Figure 5.37 presents the measured deflected shapes of tie-rod C4 at different tensile stresses in the tests with two boundary conditions. The maximum deflections are 22.5 mm and 36 mm, respectively. The effect of bending curvature due to self-weight is the most significant among the four tie-rods. Similar to tie-rod S2, when $\Delta \sigma = 0$, the tie-rod was more stressed initially in the boundary condition PP and as a result, the deflection was smaller. In addition, the tie-rod is more sensitive to changes in the tensile stress in the boundary condition FF than PP, which agrees with the results of tie-rod S2.
5.5.7.5 **Summary of the effect of bending curvature due to self-weight and the tensile stress**

Table 5-10 gives a summary of the maximum and minimum deflections in all the tests together with the tensile stress values. It can be concluded that the tie-rods are highly affected by bending curvature due to its self-weight, especially at low tensile stress. The effect of tensile stress is significant in straightening of the tie-rod. From the results of Table 5-10, for short tie-rod with the length equal or less than 2.7 m, the effect of bending curvature is reasonably acceptable. For long tie-rods with the length of 5.4 m or more, the effect of bending curvature due to self-weight is significant and should not be neglected.

Table 5-10
Summary of the maximum and minimum deflections and values of the tensile stress

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Specimen</th>
<th>Max. deflection, ( w_{\text{max}} ) (mm)</th>
<th>Min. deflection, ( w_{\text{min}} ) (mm)</th>
<th>Initial stress, ( \sigma_0 ) (MPa)</th>
<th>Max. stress, ( \sigma_3 ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PP</td>
<td>2.7 m</td>
<td>2</td>
<td>0</td>
<td>5.3</td>
<td>65.5</td>
</tr>
<tr>
<td>S1-FF</td>
<td>2.7 m</td>
<td>2</td>
<td>0</td>
<td>2.5</td>
<td>62.0</td>
</tr>
<tr>
<td>S2-PP</td>
<td>5.4 m</td>
<td>27</td>
<td>11.5</td>
<td>8.4</td>
<td>67.9</td>
</tr>
<tr>
<td>S2-FF</td>
<td>5.4 m</td>
<td>29</td>
<td>7</td>
<td>4.2</td>
<td>65.1</td>
</tr>
<tr>
<td>C3-PP</td>
<td>2.7 m</td>
<td>7</td>
<td>2.5</td>
<td>5.6</td>
<td>84.7</td>
</tr>
<tr>
<td>C3-FF</td>
<td>2.7 m</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>80.5</td>
</tr>
<tr>
<td>C4-PP</td>
<td>5.4 m</td>
<td>22.5</td>
<td>7.5</td>
<td>7.7</td>
<td>88.2</td>
</tr>
<tr>
<td>C4-FF</td>
<td>5.4 m</td>
<td>36</td>
<td>4</td>
<td>0</td>
<td>80.5</td>
</tr>
</tbody>
</table>

In the subsequent analysis, the effect of bending curvature due to self-weight will be taken into account in the numerical models of 5.4 m-long tie-rods (i.e. tie-rods S2 and C4) and tie-rod C3. For tie-rod S1, the effect is insignificant and can be neglected.

5.5.8 **Comparison of the experimental results with the numerical results including the effect of bending curvature due to self-weight**

To assess the effect of the bending curvature due to self-weight, the geometry of the models were updated with the geometry of the deflected shapes. After that, the model was re-run by the modal analysis to obtain the frequencies and modes of vibration.
5.5.8.1 Updated numerical results of tie-rod S2

The results of the updated numerical models of tie-rod S2 are shown in Figure 5.38 and Figure 5.39. Only the results for the first mode are presented because the effect is insignificant for other cases.

The effect of bending curvature due to self-weight is proven to be correct. In the boundary condition FF, the effect is less accurate. This could be due to the experimental uncertainty in the tensile stress for test S2-FF, the correct value of the initial tensile stress might be lower.
The effect of bending curvature due to self-weight is significant only for the first mode at initial low tensile stress. This agrees with the results in Section 5.5.3.

5.5.8.2 Updated numerical results of tie-rod C3

The results of the updated numerical models of tie-rod C3 are shown in Figure 5.40 and Figure 5.41.

The effect of bending curvature due to self-weight is small for tie-rod C3 as expected. Therefore, for short tie-rod with the length of less than 2.7 m, the effect can be neglected.
5.5.8.3 Updated numerical results of tie-rod C4

The results of the updated numerical models of tie-rod C4 are shown in Figure 5.42 and Figure 5.43.

![Comparison of the experimental and numerical results for tests C4-PP including the effect of bending curvature due to self-weight](image1)

![Comparison of the experimental and numerical results for tests C4-FF including the effect of bending curvature due to self-weight](image2)

Again, the effect of bending curvature due to self-weight is proven to be correct and significant for 5.4 m-long tie-rod. From all the updated results, the effect of bending curvature due to self-weight is significant only for the first mode at low tensile stress and long tie-rods. When the tie-rod is stressed more, this effect is reduced. The effect will become negligible when the tie-rod is sufficiently stressed.
5.6 Conclusion

Three types of sensors: conventional wired, wireless accelerometer and laser vibrometer were used in the experiments. The results of the the wireless accelerometer and laser vibrometer systems match accurately with that of the wired accelerometer system; the maximum error is 1.1 % and 3.2 % respectively.

The effect of bending curvature due to self-weight should be taken into account when performing the dynamic tests of metallic tie-rods and comparing with the numerical results. Depending on the length and cross-section of the tie-rod, the effect is small and might be neglected for short tie-rods, like the tie-rods S1 and C3 with the length of 2.7 m. However, for long tie-rods, like the tie-rods S2 and C4 with the length of 5.4 m, the effect is highly significant on the first mode of vibration at low values of tensile stress. Based on the experiments, low values of tensile stress could be in a range from 0 to 30 MPa.

- When the effect of bending curvature due to self-weight is insignificant:
  - The experimental results of tie-rod S1 (i.e. the shorter square tie-rod) match well with the numerical results with pinned-pinned condition. The experimental results with the boundary condition type FF are closer with the numerical results than type PP. The maximum error is 13.1% in the boundary condition type FF and 16.8% in the boundary condition type PP. This indicates that the theoretical formula of a tie-rod with pinned-pinned condition can be used to estimate the tensile force in tie-rod S1 with reasonable accuracy.
  - The experimental results of tie-rod C3 (i.e. shorter circular tie-rod) become more consistent with the numerical results with pinned-pinned condition at higher tensile stress and higher modes.

- When the effect of bending curvature due to self-weight is significant:
  - Except for the first mode at initial low tensile stress, the experimental results of tie-rod S2 approximate the numerical results with pinned-pinned condition. They follow similar trend like the tie-rod S1.
  - The experimental results of tie-rod C4 are the most affected by the effect of bending curvature due to self-weight among the four tie-rods. The results are not consistent with the numerical results with either pinned-pinned or fixed-fixed condition. However, they also have a similar trend to the results of tie-rod C3.
Overall, the results of the square tie-rods are more agreeable and consistent with the numerical results with pinned-pinned condition than the circular tie-rods. The experimental results of circular tie-rods approach the numerical results with pinned-pinned condition at high tensile stress. A preliminary conclusion is that the theoretical formula with pinned-pinned condition works well for all types of tie-rods only when the tie-rod reaches a certain stage of tensile stress. At that stage, the effect of bending curvature due to self-weight is also negligible.

As discussed in this Chapter, there might be several experimental uncertainties associated with the measured values of tensile stress. As a result, the values of the tensile stress used in the numerical models may not be precise. To better control and reduce possible errors in the measurements, a method should be implemented to measure both the applied forces and the displacements when carrying out the tests. Further analysis will be carried out in the next Chapter to assess the accuracy of the measured values of tensile stress as well as the effect of bending curvature due to self-weight.
Chapter 6

Calibration of the Numerical Models

Abstract

As carried out in Chapter 5, the modal identification tests using output-only techniques were performed for tie-rods in the laboratory. To evaluate the experimental uncertainties in the measured values of tensile stress and several other experimental parameters, the system identification analysis using numerical models will be executed in this Chapter and the results will be presented. The experimental frequencies of the tie-rods will be calibrated with the frequencies of the numerical models to find the best-suited values of several experimental parameters including the tensile stress in tie-rods. In addition, the most suitable numerical analysis method to describe accurately the dynamic behaviour of tie-rods is studied.
6.1 Introduction

The experimental uncertainties in the values of tensile stress in tie-rods might significantly affect the results of the numerical models. To simulate the true response of tie-rods at more precise values of tensile stresses, the model was calibrated using the Finite Element Model Updating (FEMU) method. This is a technique to update the modal parameters based on optimization processes that minimizes the residuals between the experimental response and the mathematical response. In simple terms, the optimization process is the minimization of the error between the experiment and the numerical modal data. The FEMU algorithm can be applied to any modal feature such as eigen frequencies, mode shape displacements, modal curvatures, etc., and their combinations. Here, only the frequencies were selected to be updated and the verification of the numerical models was based on the comparison of the experimental and numerical mode shapes. The mode shapes should be considered in the optimization process; however, due to the limited number of DoFs obtained in the experiments and time constraint to carry out the work, the mode shapes were not included in the optimization process.

The primary step of the method is to choose the variables to be updated. According to Ramos (2007), the updating parameters should influence the modal data considerably, but not all these parameters should be included in the calibration process. If a parameter can be estimated accurately in the FE model by means of experimental tests, or if the material properties of the structure are assured, there is no need to update it. Material property inaccuracy due to production process, variations in the boundary conditions and the uncertainty in the values of the tensile stress during experiments may be included in the optimization process. Following the recommendations of Ramos (2007), the selected parameters to be updated were the value of the tensile stress, the elastic modulus of the tie-rod specimen and the rotational stiffness of the supports.

After defining the variables to be updated, an initial value (i.e. base value) for each variable should be defined as well as a lower and upper bounds. These values are the boundary constraints applied to the updating parameters to obtain realistic results for the final values. At the end of the optimization process, the updated values for each selected variable are obtained. These values should be reviewed to assess whether they are reasonable and consistent based on engineering judgment and experience in the field.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

The FEMU method will be applied to all four tie-rod models using the experimental results with both types of boundary conditions PP and FF.

6.2 Mathematical model for the calibration process

The method was presented by Douglas and Reid (1982) which minimizes the difference between the theoretical and experimental dynamic parameters through a dependence of the natural frequencies (or another modal parameter) on the unknown structural variables. The resonant frequencies can be estimated from the structural variables based on Eq. (6.1):

\[ f_i^{FE} (X_1, X_2, ..., X_k) = C_j + \sum_{k=1}^{n} (A_{j,k} X_k + B_{j,k} X_k^2) \]  
(6.1)

where \( f_i^{FE} \) is the approximation of the \( j^{th} \) frequency of the FE model, \( X_k \) (\( k = 1, 2, ..., n \)) represents the unknown structural variables and \( A_{j,k} \), \( B_{j,k} \) and \( C_j \) are the calibration constants.

The established relation indicates that there are \( 2n + 1 \) constants that must be determined before any approximation. As a result, it is necessary to define the base value \( X_k^B \), the upper bound value \( X_k^U \) and the lower bound value \( X_k^L \).

After that, the constants \( A_{j,k} \), \( B_{j,k} \) and \( C_j \) are calculated by computing the \( j^{th} \) resonant frequency \( f_i^{C} \) by changing the initial values. The combination of the values of the variables follows the procedure such that the first equation is constructed by the base values of the first variable, and then the lower value and the upper value of the first parameter, and so on until all variables are utilized as following:

\[
\begin{align*}
    f_i^{FE} (X_1^B, X_2^B, ..., X_n^B) &= f_i^{C} (X_1^B, X_2^B, ..., X_n^B) \\
    f_i^{FE} (X_1^L, X_2^B, ..., X_n^B) &= f_i^{C} (X_1^L, X_2^B, ..., X_n^B) \\
    f_i^{FE} (X_1^B, X_2^L, ..., X_n^B) &= f_i^{C} (X_1^B, X_2^L, ..., X_n^B) \\
    f_i^{FE} (X_1^B, X_2^B, ..., X_n^B) &= f_i^{C} (X_1^B, X_2^B, ..., X_n^B) \\
    \vdots \\
    f_i^{FE} (X_1^B, X_2^B, ..., X_n^L) &= f_i^{C} (X_1^B, X_2^B, ..., X_n^L) \\
    f_i^{FE} (X_1^B, X_2^B, ..., X_n^U) &= f_i^{C} (X_1^B, X_2^B, ..., X_n^U) 
\end{align*}
\]  
(6.2)

where \( f_i^{C} \) is the \( i^{th} \) frequency of the numerical model in the calibration process.

After the constants in Eq. (6.1) are defined by a number of iterations in Eq. (6.2), the optimization process is completed once the objective function \( \pi \) in Eq. (6.3) is minimized.
6.3 Numerical model for the calibration process

The tie-rod model was modeled as a beam with uniform cross-section and supported by rotational springs at both ends while subjected to a constant tensile stress. The model has 20 elements and 21 nodes. The type of element is 2-node Bernoulli, isoparametric ignoring shear deformation. Different types of element can have some effects on the results of circular tie-rods. For rectangular tie-rods, this effect is not significant. To take into account the mass of both the accelerometers and the cables which is approximately 0.3 kg/each, the mass density for the numerical models is modified to 8130 kg/m³, 8060 kg/m³, 9260 kg/m³ and 8900 kg/m³ for tie-rod S1, S2, C3 and C4 respectively. The rotational spring stiffness of the supports for the fixed-fixed condition were calculated using the formula \( k = 12 \times \frac{EI}{L^3} \). These values were used as reference for the maximum values to compare with the updated results. The model is illustrated in Figure 6.1.

![Figure 6.1 – The numerical model used for the calibration process](image)

6.4 Parameters of the tie-rod models for the calibration process

Table 6-1 shows the selected variables to be calibrated and their base values. These are: (1) the tensile stress, \( \sigma \) (MPa), (2) the rotational stiffness of the supports, \( k \) (N/m) and (3) the elastic modulus, \( E \) (GPa).
Table 6-1
Base values, upper and lower bounds of the calibration variables of numerical models S1, S2, C3 and C4

<table>
<thead>
<tr>
<th>Variables</th>
<th>Parameter</th>
<th>Base value</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tensile stress, ( \sigma )</td>
<td>10</td>
<td>0</td>
<td>209</td>
<td>MPa</td>
</tr>
<tr>
<td>2</td>
<td>Rotational stiffness of the supports, ( k )</td>
<td>( \frac{1}{2} ) Max. calculated value</td>
<td>0</td>
<td>Max. calculated value</td>
<td>N/m</td>
</tr>
<tr>
<td>3</td>
<td>Elastic Modulus, ( E )</td>
<td>210</td>
<td>200</td>
<td>220</td>
<td>GPa</td>
</tr>
</tbody>
</table>

The maximum calculated value is equal to 27313 N/m, 3414 N/m, 1006 N/m and 126 N/m corresponding to tie-rod S1, S2, C3 and C4 respectively.

The optimization process was done in Matlab (2006) and DIANA (2008) programs. For the analyses of tie-rod models S1, the objective function had five eigen frequencies. For other analyses of tie-rod models S2, C3 and C4, the objective function had seven eigen frequencies. The tolerance for the residuals and the updating parameters in the objective function was equal to \( 1.0 \times 10^{-6} \) and the step increment was equal to 1%. The experimental and numerical results will be calibrated in two cases which first exclude, and then include, the effect of bending curvature due to self-weight. In the case which includes the effect of bending curvature due to self-weight, two types of analyses can be used to obtain the deflected shapes of the tie-rod models. They are linear static or geometric non-linear analysis. The geometric non-linear analysis is carried out for cables because there can be large deformations with respect to the effect of applied tensile force. Compared to cables, tie-rods take into account the bending stiffness, and the length-to-cross-section ratios are much lower. Therefore, the deformations of tie-rods are not as large as cables. To conclude, in the case which takes into account the bending curvature due to self-weight, the linear static analysis will be carried out first. The results of the linear static analysis will indicate whether it is necessary to perform the geometric non-linear analysis which accounts for large deformations.

6.5 Calibration of the tie-rod models excluding the effect of bending curvature due to self-weight

6.5.1 Calibration of tie-rod models S1 and S2 excluding the effect of bending curvature due to self-weight
The updated values of the parameters of tie-rod models S1 and S2 excluding the effect of bending curvature are shown in Table 6-2.

Table 6-2
Updated values of the parameters of tie-rod models S1 and S2 excluding the effect of bending curvature due to self-weight

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, $\text{error}_{\text{aver}}$ (%)</th>
<th>Max. freq. error, $\text{error}_{\text{max}}$ (%)</th>
<th>Error in freq. of 1$^{st}$ mode, $\text{error}_{\text{mode1}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PPa</td>
<td>5.25</td>
<td>2.21</td>
<td>-57.9</td>
<td>2290</td>
<td>210 200</td>
<td>2.5</td>
</tr>
<tr>
<td>b</td>
<td>24.9</td>
<td>9.54</td>
<td>-61.7</td>
<td>4990</td>
<td>210 200</td>
<td>5.5</td>
</tr>
<tr>
<td>c</td>
<td>45.9</td>
<td>21.4</td>
<td>-53.4</td>
<td>6110</td>
<td>210 200</td>
<td>6.8</td>
</tr>
<tr>
<td>d</td>
<td>65.5</td>
<td>34.4</td>
<td>-47.5</td>
<td>8970</td>
<td>210 200</td>
<td>2.4</td>
</tr>
<tr>
<td>S1-FFa</td>
<td>2.54</td>
<td>1.01</td>
<td>-60.2</td>
<td>2490</td>
<td>210 200</td>
<td>2.9</td>
</tr>
<tr>
<td>b</td>
<td>22.8</td>
<td>15.1</td>
<td>-33.8</td>
<td>2640</td>
<td>210 200</td>
<td>6.1</td>
</tr>
<tr>
<td>c</td>
<td>42.4</td>
<td>30.0</td>
<td>-29.2</td>
<td>3540</td>
<td>210 200</td>
<td>2.7</td>
</tr>
<tr>
<td>d</td>
<td>62.0</td>
<td>47.2</td>
<td>-23.9</td>
<td>4000</td>
<td>210 200</td>
<td>1.7</td>
</tr>
<tr>
<td>S2-PPa</td>
<td>8.40</td>
<td>13.0</td>
<td>54.8</td>
<td>1110</td>
<td>210 200</td>
<td>7.0</td>
</tr>
<tr>
<td>b</td>
<td>29.4</td>
<td>23.2</td>
<td>-21.1</td>
<td>1460</td>
<td>210 200</td>
<td>3.2</td>
</tr>
<tr>
<td>c</td>
<td>49</td>
<td>44.9</td>
<td>-8.4</td>
<td>1650</td>
<td>210 200</td>
<td>1.6</td>
</tr>
<tr>
<td>d</td>
<td>67.9</td>
<td>58.9</td>
<td>-13.3</td>
<td>2060</td>
<td>210 200</td>
<td>1.7</td>
</tr>
<tr>
<td>S2-FFa</td>
<td>4.20</td>
<td>11.7</td>
<td>178.6</td>
<td>186</td>
<td>210 200</td>
<td>2.2</td>
</tr>
<tr>
<td>b</td>
<td>24.5</td>
<td>25.2</td>
<td>2.9</td>
<td>512</td>
<td>210 200</td>
<td>1.3</td>
</tr>
<tr>
<td>c</td>
<td>44.1</td>
<td>41.4</td>
<td>-6.1</td>
<td>563</td>
<td>210 200</td>
<td>1.4</td>
</tr>
<tr>
<td>d</td>
<td>65.1</td>
<td>63.1</td>
<td>-3.1</td>
<td>939</td>
<td>210 200</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The optimization process gives reasonably good results in terms of averaged frequency errors, which are all equal or less than 7%. The maximum frequency error of separate mode is 16%. In Table 6-2, the error in the frequency of the first mode is presented in order to make the comparison between the analysis excluding or including the effect of bending curvature due to self-weight, because the analysis in Chapter 5 showed that the bending curvature due to self-weight only affects the first mode at low tensile stress. The relative frequency error of each mode at different values of tensile stresses are presented in Figure 6.2.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure 6.2 – The frequency errors of the first five or seven modes between the experimental values and final updated numerical values excluding the effect of bending curvature due to self-weight: (a) Tests S1-PP; (b) Tests S1-FF; (c) Tests S2-PP; (d) Tests S2-FF

It can be noted that the distribution of higher frequency error for tie-rod S1 focuses on higher modes. This could indicate that for short square tie-rod, the true response of the tie-rod being subjected to higher tensile stress is less sensitive compared with the numerical model or the theoretical approach when the mode is increased. The distribution of frequency error for tie-rod S2 is more uniform. The biggest error is -11.7% at the first mode in the test S1-PPa due to the effect of bending curvature, as expected. Among the four tests, tests S2-FF provide the least frequency errors, showing that the updated numerical results match almost perfectly to the experimental results.

The updated values of tensile stress for tie-rod S1 are lower than the measured values in the experiments. For tie-rod S2, they are also lower except the values of first mode at the initial tensile stress, indicating that this value is affected by the effect of bending curvature due to
self-weight. The values for tie-rod S2 are closer to the experimental values than tie-rod S1, indicating that the measurement for long tie-rods provides more accurate results. However, the relatively high difference between the tensile stresses might prove that the measurement of the tensile forces and calculated stresses in the experiments were not accurate, especially the measurement of the initial tensile force in the tie-rod by recording the values before lifting and then releasing a crane in the middle of the tie-rod. The crane which introduced a support in the middle could significantly affect the measured values of the initial displacements.

One the advantage of the FEMU method is it obtains the exact values of the rotational stiffness of the support which are unknowns in the experiments. In real situations, this would be a great advantage to obtain the rotational stiffness of ancient tie-rods clamped inside the masonry walls. During the calibration process, the values of the rotational stiffness were sensitive to the initial base value and the upper bounds. Therefore, it requires reasonable values of the upper constraints, which are the calculated maximum values for each of the four tie-rods, in order to obtain realistic results. In addition, it requires engineering judgment to evaluate the final values. The reason for the sensitive variation of the final values of the rotational stiffness can be the objective function has several close values of its minimum.

The values of the rotational stiffness are higher at higher tensile stress, as expected. They facilitate the analysis procedure as both the tensile stress and rotational stiffness of the supports have positive effects on the frequency of tie-rods. It means that they both increase the frequency of tie-rods when they are increased. Figure 6.3 presents the values of the rotational stiffness in comparison with the calculated minimum and maximum values.

![Figure 6.3 - The updated values of the rotational stiffness of the supports for tie-rods S1 and S2: (a) Tests S1-PP and S2-FF; (b) Tests S2-PP and S2-FF](image-url)
As seen in Figure 6.3, for both tie-rods S1 and S2, the values of the rotational stiffness of the support in the experimental tests with boundary condition type PP are higher than that of type FF. This can indicate the boundary conditions in the experiments were not appropriately set up to simulate the desired boundary conditions. It is possible that for the boundary condition type PP, the high friction between the female nut type A and the surface of the vertical supporting frames introduced more rotational restraints to the tie-rods than was intended. The values in the type FF are closer to the minimum value, which explains that their results are more in agreement with the numerical results with pinned-pinned condition.

The values of the rotational stiffness are divided by the maximum calculated value, and the maximum factor is 0.33 for tie-rod S1 and 0.6 for tie-rod S2. This means that for tie-rod S1, using the numerical analysis excluding the effect of bending curvature due to self-weight and when the frequency of the first mode is not considered, the frequency is in the range from \((0.8 \times f_{\text{pinned-pinned}})\) to \((0.33 \times f_{\text{fixed-fixed}})\), where \(f_{\text{pinned-pinned}}\) and \(f_{\text{fixed-fixed}}\) are the frequency of the tie-rod with pinned-pinned condition and fixed-fixed condition, respectively. The factor “0.8” before \(f_{\text{pinned-pinned}}\) accounts for the maximum positive frequency error of approximately 20%. Positive frequency errors indicate that the frequencies of the numerical models are higher than the experimental frequencies.

Similarly, the range of the frequency of tie-rod S2 is from \((0.9 \times f_{\text{pinned-pinned}})\) to \((0.6 \times f_{\text{fixed-fixed}})\). These factors can be used to estimate the tensile force based on the theoretical frequency formula for a tie-rod with pinned-pinned condition, which will be discussed in the later Section.

It can be observed that the elastic modulus \(E\) is constantly maintained as 200 GPa in all final results, which is a decrease of 5% from the initial value. However, this is an acceptable value.

**6.5.2 Calibration of tie-rod models C3 and C4 excluding the effect of bending curvature due to self-weight**

The updated values of the parameters of tie-rod models C3 and C4 are shown in Table 6-3.
Table 6-3
Updated values of the parameters of tie-rod models C3 and C4 excluding the effect of bending curvature due to self-weight

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, $\text{error}_{\text{aver}}$ (%)</th>
<th>Max. freq. error, $\text{error}_{\text{max}}$ (%)</th>
<th>Error in freq. of 1$\text{st}$ mode, $\text{error}_{\text{mode1}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C3-PPa</td>
<td>5.60</td>
<td>26.1</td>
<td>366.1</td>
<td>PP</td>
<td>2.53</td>
<td>210</td>
</tr>
<tr>
<td>b</td>
<td>25.9</td>
<td>40.2</td>
<td>55.2</td>
<td>PP</td>
<td>7.43</td>
<td>210</td>
</tr>
<tr>
<td>c</td>
<td>45.5</td>
<td>63.3</td>
<td>39.1</td>
<td>PP</td>
<td>10.0</td>
<td>210</td>
</tr>
<tr>
<td>d</td>
<td>65.8</td>
<td>77.4</td>
<td>17.6</td>
<td>PP</td>
<td>25.8</td>
<td>210</td>
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<tr>
<td>e</td>
<td>84.7</td>
<td>95.0</td>
<td>12.2</td>
<td>PP</td>
<td>33.0</td>
<td>210</td>
</tr>
<tr>
<td>C3-FFa</td>
<td>0.00</td>
<td>23.5</td>
<td>2349</td>
<td>FF</td>
<td>435</td>
<td>210</td>
</tr>
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<td>20.3</td>
<td>49.5</td>
<td>143.8</td>
<td>FF</td>
<td>500</td>
<td>210</td>
</tr>
<tr>
<td>c</td>
<td>40.6</td>
<td>71.1</td>
<td>75.1</td>
<td>FF</td>
<td>563</td>
<td>210</td>
</tr>
<tr>
<td>d</td>
<td>60.2</td>
<td>87.2</td>
<td>44.9</td>
<td>FF</td>
<td>603</td>
<td>210</td>
</tr>
<tr>
<td>e</td>
<td>80.5</td>
<td>112</td>
<td>39.1</td>
<td>FF</td>
<td>680</td>
<td>210</td>
</tr>
<tr>
<td>C4-PPa</td>
<td>7.70</td>
<td>23.7</td>
<td>207.8</td>
<td>PP</td>
<td>94.5</td>
<td>210</td>
</tr>
<tr>
<td>b</td>
<td>30.1</td>
<td>34.7</td>
<td>15.3</td>
<td>PP</td>
<td>92.8</td>
<td>210</td>
</tr>
<tr>
<td>c</td>
<td>47.6</td>
<td>49.4</td>
<td>3.8</td>
<td>PP</td>
<td>89.1</td>
<td>210</td>
</tr>
<tr>
<td>d</td>
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<td>67.7</td>
<td>-1.3</td>
<td>PP</td>
<td>89.2</td>
<td>210</td>
</tr>
<tr>
<td>e</td>
<td>88.2</td>
<td>83.4</td>
<td>-5.4</td>
<td>PP</td>
<td>89.3</td>
<td>210</td>
</tr>
<tr>
<td>C4-FFa</td>
<td>0.00</td>
<td>17.1</td>
<td>1709</td>
<td>FF</td>
<td>84.5</td>
<td>210</td>
</tr>
<tr>
<td>b</td>
<td>20.3</td>
<td>41.9</td>
<td>106.4</td>
<td>FF</td>
<td>87.4</td>
<td>210</td>
</tr>
<tr>
<td>c</td>
<td>39.2</td>
<td>60.4</td>
<td>54.1</td>
<td>FF</td>
<td>85.1</td>
<td>210</td>
</tr>
<tr>
<td>d</td>
<td>59.5</td>
<td>82.2</td>
<td>38.2</td>
<td>FF</td>
<td>91.2</td>
<td>210</td>
</tr>
<tr>
<td>e</td>
<td>80.5</td>
<td>102</td>
<td>26.7</td>
<td>FF</td>
<td>92.5</td>
<td>210</td>
</tr>
</tbody>
</table>

Similarly to tie-rod models S1 and S2, the optimization process gives reasonably good results in terms of averaged frequency error, except for the first mode at the initial value of tensile stress in the tests C4-PP and C4-FF. They are -18.7 % and -35.6 %, which are caused by the significant effect of bending curvature due to self-weight. These values will be compared with the results when the effect of bending curvature due to self-weight is included. The detailed frequency errors of each mode are presented in Figure 6.7.
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure 6.4 – The frequency errors of the first seven modes between the experimental values and final updated numerical values excluding the effect of bending curvature due to self-weight: (a) Tests C3-PP; (b) Tests C3-FF; (c) Tests C4-PP; (d) Tests C4-FF

Among the frequency errors for tie-rod C3, the values are abnormally high at the fourth mode, as also mentioned in Chapter 5. This could be due to some experimental error in the measured frequencies of this mode. Compared with tie-rod S1, the distribution of the frequency errors for tie-rod S3 does not experience higher concentration at higher modes. Therefore, the response of short circular tie-rods being subjected to different tensile stresses might follow more closely to the numerical results.

From the values of tensile stress, the updated values are much higher than the updated ones; especially the values of the initial tensile stress after calibration are extremely higher. However, it is correct that the circular tie-rods are easier to be rotated during the tests, which could affect the values of the measured displacements. The determination of the tensile stress in the long tie-rod C4 was more accurate compared to the short tie-rod C3.
Regarding the values of the rotational stiffness, the values of the tests C3-PP are very close to the pinned-pinned condition. However, the values of other tests are closer to the fixed-fixed condition. The values of rotational stiffness are compared with the calculated minimum and maximum values in Figure 6.5.

![Figure 6.5 - The updated values of the rotational stiffness of the supports for tie-rods C3 and C4: (a) Tests C3-PP and C3-FF; (b) Tests C4-PP and C4-FF](image)

The values of the rotational stiffness are divided by the maximum calculated value. The maximum factor is 0.68 for tie-rod C3 and 0.75 for tie-rod C4. Therefore, for tie-rod C3, using the numerical analysis excluding the effect of bending curvature due to self-weight, the frequency is in the range from \((0.8 \times f_{pinned-pinned})\) to \((0.68 \times f_{fixed-fixed})\). For tie-rod C4, the frequency is in the range from \((0.9 \times f_{pinned-pinned})\) to \((0.75 \times f_{fixed-fixed})\).

### 6.6 Calibration of tie-rod model including the effect of bending curvature due to self-weight using linear analysis

To take into account the effect of bending curvature due to self-weight in the calibration process, the model is first analyzed by linear static analysis. Only the self-weight of the tie-rod, the weight of the accelerometers and an applied tensile stress are considered. From the linear static analysis, the deformed shape of the tie-rod is obtained, which corresponds to the value of the initial applied tensile stress. The geometry of the model was updated with the geometry of the deformed shape. After that, the calibration process begins with a value set of the updating parameters, in which the tensile stress is kept the same. The model is run by the modal analysis to obtain the frequencies and modes of vibration.
Each time when the tensile stress, as one of the parameters, is updated, the model is re-run by linear static analysis to obtain a new deflected shape. The geometry of the model is updated with this new deflected shape and the calibration process continues. This calibration process including the effect of bending curvature due to self-weight using linear static analysis was also written in Matlab (2006) and DIANA (2008) programs.

### 6.6.1 Calibration of tie-rod models S1 and S2 including the effect of bending curvature due to self-weight using linear analysis

The updated values of the parameters of tie-rod models S1 and S2 including the effect of bending curvature due to self-weight using linear analysis are shown in Table 6-4.

#### Table 6-4

Updated values of the parameters of tie-rod models S1 and S2 including the effect of bending curvature due to self-weight using linear analysis

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, $\overline{\text{error}}$ (%)</th>
<th>Max. freq. error, $\text{error}_{\text{max}}$ (%)</th>
<th>Error in freq. of 1$^\text{st}$ mode, $\text{error}_{\text{model}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PP</td>
<td>5.25</td>
<td>0.87</td>
<td>-83.4</td>
<td>PP</td>
<td>2270</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>24.9</td>
<td>12.3</td>
<td>-50.6</td>
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<tr>
<td></td>
<td>45.9</td>
<td>26.4</td>
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<td>PP</td>
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<td>210</td>
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<td></td>
<td>65.5</td>
<td>43.7</td>
<td>-33.3</td>
<td>PP</td>
<td>4060</td>
<td>210</td>
</tr>
<tr>
<td>S1-FF</td>
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</tr>
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<td></td>
<td>22.8</td>
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<td>-43.9</td>
<td>FF</td>
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<td>42.4</td>
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<td>-26.9</td>
<td>FF</td>
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<td>FF</td>
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<td>S2-PP</td>
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<td>PP</td>
<td>1090</td>
<td>210</td>
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<tr>
<td></td>
<td>29.4</td>
<td>10.0</td>
<td>-66.0</td>
<td>PP</td>
<td>2420</td>
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</tr>
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<td></td>
<td>49.0</td>
<td>44.1</td>
<td>-10.0</td>
<td>PP</td>
<td>2380</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>67.9</td>
<td>59.6</td>
<td>-12.2</td>
<td>PP</td>
<td>3090</td>
<td>210</td>
</tr>
<tr>
<td>S2-FF</td>
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<td>5.56</td>
<td>32.4</td>
<td>FF</td>
<td>2280</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>24.5</td>
<td>10.7</td>
<td>-56.3</td>
<td>FF</td>
<td>3210</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>44.1</td>
<td>32.7</td>
<td>-25.9</td>
<td>FF</td>
<td>3330</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>65.1</td>
<td>64.7</td>
<td>-0.6</td>
<td>FF</td>
<td>3410</td>
<td>210</td>
</tr>
</tbody>
</table>
Chapter 6: Calibration of the numerical models

The results are not much different for tie-rod S1 because the effect of bending curvature due to self-weight for this tie-rod is insignificant. However, for tie-rod S2, the results are significantly affected at the initial values of tensile stress in both tests S2-PP and S2-FF. The frequency errors in these cases are reduced from 11.7 % to 2.3 % and 2.3 % to 0.5 %. Figure 6.6 presents the detailed frequency errors of each mode.

![Graphs showing frequency errors for different modes and tests](image)

Figure 6.6 – The frequency errors of the first five or seven modes between the experimental values and final updated numerical values including the effect of bending curvature due to self-weight using linear analysis: (a) Tests S1-PP; (b) Tests S1-FF; (c) Tests S2-PP; (d) Tests S2-FF

It can be seen in Figure 6.6c, compared with Figure 6.2c, there is a significant reduction in the frequency error of the first mode at initial tensile stress for tie-rod S2. The distribution of the frequency errors in all tests concentrates on higher modes. On the other hand, this analysis gives consistently accurate results for lower modes.

Comparing the updated values of the tensile stress in Table 6-4 with the values in the previous analysis excluding the bending curvature due to self-weight (in Table 6-2), it could be noted that the initial tensile stresses are lower, as expected. This is because a deflected tie-
rod has higher frequency compared to a straight one. The analysis including the bending curvature due to self-weight using linear method provides better results for tie-rods S1 and S2.

### 6.6.2 Calibration of tie-rod models C3 and C4 including the effect of bending curvature due to self-weight using linear analysis

The updated values of the parameters of tie-rod models S1 and S2 including the effect of bending curvature due to self-weight using linear analysis are shown in Table 6-5.

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, $\text{error}_{\text{aver}}$ (%)</th>
<th>Max. freq. error, $\text{error}_{\text{max}}$ (%)</th>
<th>Error in freq. of 1st mode, $\text{error}_{\text{mode}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C3-PPa b</td>
<td>5.60 21.9 291.1 PP 189</td>
<td>210 200</td>
<td>2.3 5.5 0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>45.9 34.3 27.7 PP 378</td>
<td>210 200</td>
<td>1.8 5.2 0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>65.8 75.6 14.9 PP 355</td>
<td>210 200</td>
<td>2.2 5.3 -0.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>84.7 97.3 14.9 PP 399</td>
<td>210 200</td>
<td>1.9 5.5 -1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C3-FFa b</td>
<td>20.3 50.0 146.3 FF 239</td>
<td>210 200</td>
<td>3.4 10.6 -0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>40.6 64.4 58.6 FF 473</td>
<td>210 200</td>
<td>3.4 11.1 -2.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>60.2 100 66.1 FF 359</td>
<td>210 200</td>
<td>3.4 9.4 -2.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>80.5 116 44.1 FF 508</td>
<td>210 200</td>
<td>3.0 11.1 -2.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C4-PPa b</td>
<td>7.70 10.0 29.9 PP 80</td>
<td>210 200</td>
<td>9.2 28.8 28.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>30.1 46.6 54.8 PP 93</td>
<td>210 200</td>
<td>13.5 25.8 -13.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>47.6 200 320.2 PP 112</td>
<td>210 220</td>
<td>9.0 19.3 12.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>68.6 200 191.5 PP 108</td>
<td>210 220</td>
<td>12.3 27.0 1.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C4-FFa b</td>
<td>0.00 0.00 0.0 FF 71</td>
<td>210 206</td>
<td>4.5 -9.8 5.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>20.3 78.5 286.7 FF 101</td>
<td>210 200</td>
<td>14.7 27.7 27.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>39.2 198 405.1 FF 126</td>
<td>210 89</td>
<td>9.8 18.7 18.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>59.5 200 236.1 FF 126</td>
<td>210 104</td>
<td>19.0 -31.3 1.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Similarly, the updated results for tie-rod C3 do not vary much compared with the previous analysis, proving that the effect of bending curvature due to self-weight is not significant for short tie-rods. However, the updated results of tie-rod C4 are not improved like expected. Furthermore, the averaged and maximum frequency errors are remarkably high except in test C4-FFa. The detailed results of frequency errors of each mode are presented in Figure 6.7.

![Frequency Error Graphs](image)

Figure 6.7 – The frequency errors of the first seven modes between the experimental values and final updated numerical values including the effect of bending curvature due to self-weight using linear analysis: (a) Tests C3-PP; (b) Tests C3-FF; (c) Tests C4-PP; (d) Tests C4-FF

It can be noted that not only the frequency errors of all modes are very high but also the updated values of tensile stress are not reasonable for tie-rod C4. This can be explained in the next Section by comparing the deflected shapes obtained in the updated numerical models with the measured deflected shapes in the experiments.
6.6.3 Comparison of the measured experimental and updated numerical deflected shapes of tie-rods using linear analysis

This Section is important to evaluate the updated results of the numerical models using the analysis which includes the effect of bending curvature in Sections 6.6.2 and 6.6.3. As described previously, the geometry of the models was updated with the deflected shapes due to self-weight, weight of the accelerometers and cables and also the straightening effect of different applied tensile stresses. The final deflected shapes should be compared with the measured shapes in the experiments to assess whether the results are reliable.

Figure 6.8 and Figure 6.9 show the measured deflected shapes and the deflected shapes obtained by the optimization process at different tensile stresses of tie-rod S1.

Figure 6.8 – Comparison between measured and updated numerical deflected shapes in tests S1-PP

Figure 6.9 – Comparison between measured and updated numerical deflected shapes in tests S1-FF
The maximum deflections are close. However, when the tensile stress is increased, the numerical deflected shapes do not vary as significantly as observed in the experiments; although the results of the updated models of tie-rod S1 are not really affected because all the deflections are small. The effect of tensile stress can be seen more clearly in the following figures for other tie-rods. In Figure 6.10 and Figure 6.11, the measured and numerical deflected shapes of tie-rod S2 at different tensile stresses are presented.
The results show some differences in the maximum deflections, which seem to be unreasonable considering that the updated tensile stresses are lower than the measured ones. However, this could be explained by the high values of the rotational stiffness obtained in the optimization process. Similarly, Figure 6.12 and Figure 6.13 present the measured and the updated deflected shapes of tie-rod C3.

Figure 6.12 – Comparison between measured experimental and updated numerical deflected shapes of tie-rod C3 at different tensile stress in tests C3-PP

Figure 6.13 – Comparison between measured experimental and updated numerical deflected shapes of tie-rod C3 at different tensile stress in tests C3-FF
It can be seen that the differences increase for circular tie-rods. However, for tie-rod C3, the deflections are not large either and therefore, the effect on the results of this tie-rod is insignificant. The results expect to be evaluated the most accurate for long circular tie-rod C4, as its deflections are the largest. Figure 6.14 and Figure 6.15 show the measured and updated deflected shapes at different tensile stresses of tie-rod C4.

Figure 6.14 – Comparison between measured experimental and updated numerical deflected shapes of tie-rod C4 at different tensile stress in the tests C4-PP

Figure 6.15 – Comparison between measured experimental and updated numerical deflected shapes of tie-rod C4 at different tensile stress in the tests C4-FF
As expected, among the four tie-rods, the updated deflected shapes of tie-rod S4 show the most significant differences when the tensile stresses are increased. This explains the high frequency errors in Section 6.6.2. The results of all the deflected shapes prove that the linear static analysis is not suitable to obtain the deflections of tie-rods at higher tensile stresses, because the straightening effect due to the tensile stress is not sufficiently reflected. The tie-rods subject to relatively large deformations due to the effect of tensile stress and therefore, the geometric non-linear analysis which accounts for large deformations should be used.

6.6.4 Summary of the calibration results including the effect of bending curvature due to self-weight using linear analysis

A summary of the measured and updated numerical maximum and minimum deflections using linear static analysis is given in Table 6-6, together with the values of tensile stress.

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Specimen</th>
<th>Max. deflection, ( w_{\text{max}} ) (mm)</th>
<th>Min. deflection, ( w_{\text{min}} ) (mm)</th>
<th>Min. stress, ( \sigma_{\text{min}} ) (MPa)</th>
<th>Max. stress, ( \sigma_{\text{max}} ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-PP</td>
<td>2.7 m</td>
<td>2</td>
<td>2.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1-FF</td>
<td>2.7 m</td>
<td>2</td>
<td>2.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2-PP</td>
<td>5.4 m</td>
<td>27</td>
<td>22</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>S2-FF</td>
<td>5.4 m</td>
<td>29</td>
<td>21</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>C3-PP</td>
<td>2.7 m</td>
<td>7</td>
<td>6.5</td>
<td>2.5</td>
<td>4.9</td>
</tr>
<tr>
<td>C3-FF</td>
<td>2.7 m</td>
<td>3</td>
<td>6.2</td>
<td>1</td>
<td>4.5</td>
</tr>
<tr>
<td>C4-PP</td>
<td>5.4 m</td>
<td>23</td>
<td>40</td>
<td>7.5</td>
<td>34</td>
</tr>
<tr>
<td>C4-FF</td>
<td>5.4 m</td>
<td>36</td>
<td>54</td>
<td>4</td>
<td>36</td>
</tr>
</tbody>
</table>

The deflections are reasonably close, except for that of tie-rod C4. Based on the four tie-rod specimens, the linear static analysis can be used when the length-to-cross-section ratios of the tie-rods are not as high as that of tie-rod C4. Compared to cable elements, tie-rods experience smaller deformations because of the bending stiffness. Compared to beam elements, tie-rods have much larger deformations due to high slenderness ratio. The behavior of a tie-rod is between cable element and beam element, although it is closer to cable behavior.
In Table 6-6, for rectangular tie-rods, the values of the tensile stresses are lower, whereas they are higher for circular tie-rods. Again, the maximum values of tensile stress for tie-rod C4 are unreasonable and should not be considered because of the vast differences in the deflections. The significant differences in the measured and updated values of tensile stress, especially for circular tie-rods, suggest that a method must be implemented to measure the applied tensile force in the experiments to compare with the results of the calculated tensile stress obtained from the displacements of the LVDTs. One of the methods to measure the applied tensile force is by using the load cell.

6.7 Calibration of tie-rod model including the effect of bending curvature due to self-weight using geometric non-linear analysis

The calibration process is similar to that using the linear static analysis, except that the deflected shapes were obtained by the geometric non-linear analysis. In addition, instead of applying the tensile force like in the linear static analysis, the effect of tensile stress is considered by applying a displacement in the horizontal direction at the end node of the model. The applied displacement has the same effect as an applied horizontal force but it has an advantage of maintaining the type of support condition. If the horizontal force is applied, the restraint in the horizontal direction at one support must be released. During the calibration process, the value of the applied displacement is controlled by the equation: \( \Delta L = \varepsilon x L_0 = (\sigma x L_0)/E \). As a result, the applied displacement is related to the two updating parameters, i.e. the tensile stress and the elastic modulus. Every time one of these two parameters is updated, the geometric non-linear analysis is re-run to obtain a new deflected shape and the calibration process continues.

6.7.1 Calibration of tie-rod model C4 including the effect of bending curvature due to self-weight using geometric non-linear analysis

The analysis including the effect of bending curvature due to self-weight using geometric non-linear analysis was only carried out for tie-rod models C4 because the effect of bending curvature is the most significant for this tie-rod. In addition, there was limited time to carry out the work. The updated values of the parameters for several selected cases including the effect of bending curvature due to self-weight using geometric non-linear are shown in Table 6-7.
Table 6-7
Updated values of the parameters of tie-rod model C4 including the effect of bending curvature due to self-weight using geometrical non-linear analysis

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, erroraver (%)</th>
<th>Max. freq. error, errormax (%)</th>
<th>Error in freq. of 1st mode, errormode (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>Updated Num.</td>
<td>Diff. (%)</td>
<td>Exp.</td>
<td>Updated Num.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C4-PPa</td>
<td>7.70</td>
<td>15.8</td>
<td>29.9</td>
<td>PP</td>
<td>76</td>
<td>210</td>
</tr>
<tr>
<td>c</td>
<td>47.6</td>
<td>51.9</td>
<td>320.2</td>
<td>PP</td>
<td>104</td>
<td>210</td>
</tr>
<tr>
<td>C4-FFa</td>
<td>0.00</td>
<td>9.5</td>
<td>0.0</td>
<td>FF</td>
<td>63</td>
<td>210</td>
</tr>
<tr>
<td>b</td>
<td>20.3</td>
<td>47.1</td>
<td>286.7</td>
<td>FF</td>
<td>89</td>
<td>210</td>
</tr>
</tbody>
</table>

The calibration for the tests C4-PPc and C4-FFb was done to examine the changes in the deflections at higher tensile stress. Compared to the analysis using linear analysis, the averaged frequency errors are reduced in all tests except in the test C4-FFa. The frequency error of the first mode in the test C4-PPa is reduced from 28.8% when using linear analysis to 15.9%. However, in the test C4-FFa, the frequency of the first mode is relatively high. The updated deflected shapes are compared with the measured ones in Figure 6.16.

![Graph](image)

Figure 6.16 – Comparison between measured experimental and updated geometrical non-linear numerical deflections at two values of tensile stress for tie-rod C4

It can be seen that the deflections using the geometric non-linear analysis are smaller than the measured deflections. This could be due to the several factors related to the numerical models such as the type of elements used, the number of elements for the FE analysis, the
method of applying displacement using the equation relating to the values of tensile stress, etc. Among the factors, the results are expected to be more accurate if the number of elements is increased.

Compared to linear analysis, the changes in the displacements at higher tensile stress are significant and much more accurately reflected.

It is noted that in all analyses for tie-rod C4 and almost all cases, the frequency errors of the first mode are considerably higher than other modes because the effect of bending curvature is significant for the first mode at low tensile stress. Therefore, further analysis was carried out which excludes the frequency of the first mode in the calibration process. The results for two selected cases are shown in Table 6-8.

Table 6-8
Updated values of the parameters of tie-rod model C4 including the effect of bending curvature due to self-weight using geometrical non-linear analysis without calibrating the frequency of the 1st mode

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Tensile stress, $\sigma$ (MPa)</th>
<th>Rotational stiffness, $k$ (N/m)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Average freq. error, $\text{error}_{\text{aver}}$ (%)</th>
<th>Max. freq. error, $\text{error}_{\text{max}}$ (%)</th>
<th>Error in freq. of 1st mode, $\text{error}_{\text{mode1}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>Updated</td>
<td>Diff. (%)</td>
<td>Exp.</td>
<td>Updated</td>
<td>Exp.</td>
<td>Updated</td>
</tr>
<tr>
<td>C4-PPa</td>
<td>7.70</td>
<td>24.1</td>
<td>29.9</td>
<td>PP</td>
<td>47</td>
<td>210</td>
</tr>
<tr>
<td>C4-FFa</td>
<td>0.00</td>
<td>12.4</td>
<td>0.0</td>
<td>FF</td>
<td>51</td>
<td>210</td>
</tr>
</tbody>
</table>

As expected, the averaged frequency errors are significantly reduced, however, the updated values of tensile stress are higher.

6.8 Comparison of the calibration results using different methods

Table 6-9 gives a summary of the experimental and updated numerical results of all different analysis methods for tie-rod C4. The six different analyses are:

- Analysis 1: Excluding bending curvature due to self-weight; with 1st mode;
- Analysis 2: Excluding bending curvature due to self-weight; without 1st mode;
- Analysis 3: Including bending curvature due to self-weight, linear static, with 1st mode;
- Analysis 4: Including bending curvature due to self-weight, linear static, without 1st mode;
- Analysis 5: Including bending curvature due to self-weight, geometric non-linear, with 1st mode;
- Analysis 6: Including bending curvature due to self-weight, geometric non-linear, without 1st mode.
Table 6-9
Summary of the experimental results and updated numerical results using different analysis methods

<table>
<thead>
<tr>
<th>Test no.</th>
<th>Exp. and Analysis method</th>
<th>Max. deflec. (mm)</th>
<th>Diff. (%)</th>
<th>σ (MPa)</th>
<th>Diff. (%)</th>
<th>k (N/m)</th>
<th>E (GPa)</th>
<th>Aver. Error</th>
<th>Max. error in 1st mode</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>C4-PPa</td>
<td>Exp. 23</td>
<td>7.70</td>
<td>PP</td>
<td>210</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 1 0.0 -100</td>
<td>23.0</td>
<td>199</td>
<td>95</td>
<td>200</td>
<td>5.1</td>
<td>-18.7</td>
<td>-18.7</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>Anal. 2 0.0 -100</td>
<td>19.4</td>
<td>152</td>
<td>91</td>
<td>204</td>
<td>1.6</td>
<td>3.30</td>
<td>–</td>
<td>17</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 3 40 74</td>
<td>10.0</td>
<td>30.0</td>
<td>80</td>
<td>200</td>
<td>9.2</td>
<td>28.8</td>
<td>28.8</td>
<td>15</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 4 46 100</td>
<td>106</td>
<td>1277</td>
<td>118</td>
<td>200</td>
<td>2.7</td>
<td>-4.40</td>
<td>–</td>
<td>17</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 5 15 -35</td>
<td>15.8</td>
<td>105</td>
<td>76</td>
<td>200</td>
<td>5.9</td>
<td>-15.9</td>
<td>-15.9</td>
<td>32</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 6 12 -48</td>
<td>24.1</td>
<td>213</td>
<td>57</td>
<td>200</td>
<td>1.4</td>
<td>-1.90</td>
<td>–</td>
<td>38</td>
<td>–</td>
</tr>
<tr>
<td>C4-FFa</td>
<td>Exp. 36</td>
<td>0.00</td>
<td>FF</td>
<td>210</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 1 0.0 -100</td>
<td>17.1</td>
<td>1610</td>
<td>85</td>
<td>200</td>
<td>7.6</td>
<td>-35.6</td>
<td>-35.6</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>Anal. 2 0.0 -100</td>
<td>13.8</td>
<td>1280</td>
<td>83</td>
<td>206</td>
<td>1.6</td>
<td>-2.80</td>
<td>–</td>
<td>17</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 3 54 50</td>
<td>0.01</td>
<td>0.00</td>
<td>71</td>
<td>200</td>
<td>4.5</td>
<td>-9.80</td>
<td>5.20</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 4 47 31</td>
<td>21.9</td>
<td>2090</td>
<td>113</td>
<td>200</td>
<td>2.4</td>
<td>-4.60</td>
<td>–</td>
<td>11</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 5 26 -28</td>
<td>9.50</td>
<td>850</td>
<td>63</td>
<td>200</td>
<td>6.2</td>
<td>-16.7</td>
<td>-16.7</td>
<td>12</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Anal. 6 14 -61</td>
<td>12.4</td>
<td>1140</td>
<td>51</td>
<td>200</td>
<td>1.3</td>
<td>3.00</td>
<td>–</td>
<td>35</td>
<td>–</td>
</tr>
</tbody>
</table>

The differences were calculated by dividing the deflections and the tensile stresses of the analysis methods by the measured values in the experiments. The results must be compared in combination with the deflections shown in the table as well as in Figure 6.17.

Figure 6.17 – Comparison between measured experimental deflections and updated numerical deflections using different analysis methods
It can be seen that in terms of deflection values, the analysis method 5 provides the closest results to measured values. In terms of tensile stress, the analysis method 3 gives the closest results, followed by the analysis method 5. However, the analysis method 3 is not suitable when the tensile stress is increased as the change in the deflection is not significant (see Section 6.6.3). It should also be noted that the tensile stress determined from the measured tensile force might not be accurate due to several experimental uncertainties. The analysis method 4 gives the most different tensile stress, followed by the methods 1, 6 and 2. The values of the rotational stiffness are relatively difficult to assess the accuracy level of different analysis methods. In terms of averaged frequency errors, all analysis methods show acceptable results. In terms of maximum frequency errors, it is certain that the analysis methods 2, 4 and 6 give the smallest values because they exclude the frequency of first mode but on the other hand, the analysis methods 4 and 6 provide unreasonable results of the tensile stress. Overall, it is possible to conclude that the analysis method 5 is the better method in almost all cases. In other words, the analysis including the effect of bending curvature due to self-weight using geometric non-linear analysis is the most suitable method for tie-rods. However, further work should be carried out to improve this method to get closer deflections and reduce the frequency error of the first mode which is still relatively high in the test C4-FFa.

For tie-rods which are affected significantly by the effect of bending curvature due to self-weight, the analysis method 5 must be used. To be conservative, the analysis method 2 can also be used when the effect of bending stiffness is significant although it will provide higher value of tensile stress. When the effect of bending curvature is insignificant, the simplest method (i.e. analysis method 1) or the methods 2 and 3 can be used with reasonable accuracy.

In conclusion, in all cases, the analysis methods 2 and 5 can be used. The difference between the two methods is method 2 will provide more conservative results in terms of tensile stress in tie-rods whereas the method 5 will be likely to give more accurate results. Moreover, the method 2 will not give the deflections of tie-rods.

### 6.9 Revision of the estimation of tensile force/stress in tie-rods using theoretical equation and proposed standard charts

As discussed in Chapter 4, the numerical-based methodology to estimate a range of the tensile force/stress in tie-rod is by using the theoretical equation for a tie-rod with pinned-pinned condition as follows
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

\[
T = \frac{4mL^2 f_n^2}{n^2} - \frac{n^2 \pi^2 EI}{L^2}
\]

where \(2.267 f_n(pinned-pinned) = f_n(fixed-fixed)\). Using the Eq. (6.5), the tie-rod is assumed to be straight, therefore, the effect of bending curvature due to self-weight is not considered. As a result, the frequency of the first mode should not be used.

Based on the results of the calibration process in Sections 6.5.1 and 6.5.2, the above theoretical formula can be used because the frequency errors are relatively small except for the that of the first mode. The range of the frequency obtained from the calibration process is

\[
0.8 f_n(pinned-pinned) \leq f_n \leq 0.75 f_n(fixed-fixed)
\]

Combining Eqs. (6.5) and (6.6) and using the most conservative range of the frequency, one gets

\[
T = \frac{4mL^2 f_n^2}{n^2} - \frac{n^2 \pi^2 EI}{L^2}
\]

in which the frequency of the first mode is neglected.

As a result of Eq. (6.7), the first proposed standard chart I-a in Chapter 4 is updated by multiplying the frequency of pinned-pinned condition with 0.8. The frequency of the first mode should be neglected.

The correction factors for the values of \(h/L\) ratio and frequency \(f\) must be used for the standard charts:

Rectangular cross-section

\[
\left(\frac{h}{L}\right)_{corrected} = [n] \left(\frac{h}{L}\right)
\]

\[
f_n_{corrected} = \left(\frac{1}{n}\right) L f_n
\]

Circular cross-section

\[
\left(\frac{h}{L}\right)_{corrected} = [0.8660(n)] \left(\frac{h}{L}\right)
\]

\[
f_n_{corrected} = 0.9098 \left(\frac{1}{n}\right) L f_n
\]

where \(h\) is the height of the rectangular tie-rod or the diameter of the circular tie-rod (m), \(n\) is the mode number, \(L\) is the length of the tie-rod (m) and \(f_n\) is the frequency of mode \(n^{th}\) (Hz).
The two updated proposed standard charts II-a and II-b are presented in Figure 6.18. The chart for fixed-pinned condition is not included.

Figure 6.18 – The updated proposed standard charts: (a) Chart II-a; (b) Chart II-b

The Chart IIa provides the minimum value of the estimated tensile stress, whereas the Chart IIb gives the maximum value. A zoom of the chart II-b at low tensile stresses is given in Appendix A.3.
6.10 Conclusion

The results of the calibration process using the optimization technique show the significant differences between the updated values of the tensile stress and the measured ones in the experiments. This suggests that a method such as load cell should be implemented in the experiments to measure the applied force and compare with the measured displacements of the LVDTs.

The effect of bending curvature should be taken into account in the numerical simulation and modal analysis of tie-rods. For long tie-rods at low values of applied tensile stress, this effect cannot be neglected. At low tensile stress, the tie-rod experiences certain behavior which is similar to the catenary cable. However, it cannot be analyzed like a cable because it is necessary to account for bending stiffness. When it is subjected to increasing tensile stresses, large displacements occur. Therefore, the analysis should be done as geometric nonlinear analysis to obtain the deflected shapes of the tie-rods. After that, the geometry of the numerical model should be updated with the deflected shape and modal analysis can be performed to obtain the frequencies and modes of vibration.

Despite the efforts to carry out different types of analysis, the analysis which reflects the exact dynamic behavior of tie-rods has not been perfectly achieved. The analysis method including the effect of bending curvature due to self-weight using geometric non-linear analysis provides reasonable and relatively accurate results, although further improvement should be made. The results obtained in this works also provide the foundation for the technique to identify in-situ the tensile force in tie-rods taking into account the effect of bending curvature due to self-weight, which will be discussed in Chapter 7.
Chapter 7

Conclusion and Recommendation
7.1 Conclusions

A study was carried out to develop a methodology for experimental estimation of service stress levels in tie-rods in historical constructions using modal analysis tests as ND tests. At first, a compilation of a state of the art of ancient tie-rods was studied, in order to determine the common characteristics of ancient tie-rods. Then, a numerical model was developed for axially loaded tie-rods with uniform cross-section and supported by rotational springs at both ends. A parametric study was carried out with a total of 432 analyses of different bending stiffnesses and lengths, considering different applied tensile stresses and boundary conditions. Based on the results of the parametric study, a methodology and several standard charts were proposed to determine a range of tensile stress of tie-rods with different characteristics.

To validate the results of the numerical-based methodology and charts, experimental modal analysis tests were performed for four specimens in the laboratory. In total, there were 84 tests that were carried out. For comparison, three different types of sensors, conventional, wireless and laser vibrometer were used. Then, the results of the experiments were calibrated with the numerical results using the optimization technique. The tensile stress, rotational stiffness of the supports and elastic modulus of tie-rods were determined in the calibration. They were compared again with the experimental results to assess several types of experimental uncertainties. In addition, the most suitable analysis method which could be used to describe the exact dynamic response of tie-rods was studied. Finally, the methodology to estimate a range of tensile stress in tie-rods was concluded and techniques to identify in-situ the tensile stress in tie-rods was discussed.

From the results of the parametric study, the tensile stress affects both the vibration modes and frequency of tie-rods. The level of the effect depends on the length, mode number and boundary condition. The effect is more significant for longer lengths, lower modes and fixed-fixed end condition. The maximum effect of tensile stress on the frequency is $+\sqrt{x_\sigma}$, where $x_\sigma$ is the ratio between the new and original tensile stress. Beside the tensile stress, the factors that affect the frequency of tie-rods are the mode number, length, size and shape of cross-section and boundary condition. Similarly, the corresponding maximum effects are $+x_{\text{mode}}^2$, $-1/x_L^2$, $+x_h$, $-0.867$ (when $h = d$) and $+2.267$, where $h$ is the height of rectangular tie-rods and $d$ is the diameter of circular tie-rods. The “+” and “-” signs indicate the positive and negative
effects, respectively. The estimation of tensile stress in tie-rods can be done using the theoretical frequency formula or several proposed standard charts.

From the experiments, the results of the the wireless accelerometer and laser vibrometer systems match accurately with that of the wired accelerometer system; the maximum error is 1.1% and 3.2%, respectively. It was noted that the effect of bending curvature due to self-weight should be taken into account when performing the dynamic tests of metallic tie-rods and comparing with the numerical results. Depending on the length and cross-section of the tie-rod, the effect is small and might be neglected for short tie-rods with the length of 2.7 m or less. However, for long tie-rods with the length of 5.4 m or more, the effect is highly significant on the first mode of vibration at low values of tensile stress. Based on the experiments, low values of tensile stress could be in a range from 0 to 30 MPa. Overall, the results of the square tie-rods are more agreeable and consistent with the numerical results with pinned-pinned end condition than the circular tie-rods.

From the results of the calibration process, there were significant differences between the updated values of the tensile stress and the measured ones in the experiments. Regarding the effect of bending curvature due to self-weight, it was proven that the effect cannot be neglected for long tie-rods at low tensile stresses. The numerical analysis method that is the most suitable to describe the behavior of tie-rods taking into account the effect of bending curvature due to self-weight is the geometric non-linear analysis which accounts for large deformations. From the geometric non-linear analysis, the deflected shapes of tie-rods can be obtained due to self-weight and straightening effect of tensile stress. The geometry of the tie-rod used in the modal analysis should be the obtained deflected shapes. When the deflections are not interested, the analysis which excludes the effect of bending curvature (i.e. assumes straight tie-rods) can be used, however, the frequency of the first mode should not be considered. This analysis provides relatively accurate and more conservative results because it gives higher values of tensile stress.

7.1.1 Estimation of a range of tensile force/stress in tie-rods

- Preliminary estimation of a range of tensile force/ tensile stress in tie-rods using formula

Based on the theoretical frequency formula of a tie-rod with pinned-pinned condition combined with the results of the numerical parametric study and calibration process, a range of tensile force in tie-rod can be estimated by the following equation
Chapter 7: Conclusion and recommendation

\[
T = \frac{4\bar{m}L^2 f_n^2}{n^2} - \frac{n^2 \pi^2 EI}{L^2}
\]

\[
0.8 \frac{f_{n(pinned-pinned)}}{f_n} \leq f_n \leq 2.267 \frac{f_{n(pinned-pinned)}}{f_n}
\]

where $T$ is the tensile force (N), $f_n$ is the frequency of mode $n^{th}$ (Hz), $L$ is the length of the tie-rod (m), $EI$ is the bending stiffness (Nm²) and $\bar{m}$ is mass per unit length of the tie-rod (kg/m). The frequency of the first mode should not be considered.

- Preliminary estimation of a range of tensile force/ tensile stress in tie-rods using two standard charts

The construction of the standard charts is based on the combined effect of different factors using the results of the theoretical approach and numerical parametric study of 432 analyses. The two proposed standard chart II-a and II-b can be used, provided that the correction factors must be used and the frequency of the first mode is excluded. Chart II-a gives the minimum estimated value of tensile stress and chart II-b gives the maximum estimated value. The charts are provided in Section 6.10.

7.2 Recommendations

7.2.1 Recommendation for the performance of modal analysis tests of tie-rods

The number of sensors used and their positions are determined based on the first several modes of vibration of tie-rods. To acquire more accurate results, the number of modes for tie-rods is from 5 to 8 modes. The number of sensors should be sufficient to obtain the mode shapes, and their positions should be determined to avoid the locations of zero displacement for each mode of vibration.

Generally, the Frequency Response Spectrum (FRS) of tie-rods has clear peaks, so that the frequencies can be determined without difficulty. The higher the number of sensors, the clearer the FRS obtained. If only the frequencies are interested not the mode shapes, even one sensor can be used for simplification of the tests. However, the mode shapes should be obtained in the tests for verification of the results.

All three types of sensors: conventional wired accelerometer, wireless accelerometer and laser vibrometer can be used. The results are close. The disadvantages of the wired accelerometer are the cable and the additional weight on the tie-rod, which can be significant if many accelerometers are used and/or the tie-rod has small cross-section. They can also have some effects on the dynamic behavior of tie-rod although the effect is not so significant. The
wireless accelerometer and laser vibrometer can avoid those disadvantages. However, the wireless accelerometer can only acquire the low frequency range and has more time-consuming testing procedure. The laser vibrometer requires focusing the sensor heads to obtain full signals. Overall, the laser vibrometer system is recommended.

To measure the tensile stress in tie-rods in the tests, two measurements should be used: one to measure the applied tensile force using the load cell equipment, and the other one to measure the displacement in tie-rods using the LVDTs. At least two LVDTs should be used at one location. Two LVDTs should be placed near one end of the tie-rod, two in the middle and two near the other end. The higher numbers of the LVDTs at different locations, the better the tests are to compare the measurements. At one location, the two LVDTs should be placed on two sides of the tie-rod to take the average value. They should be placed parallel to and at the same level as the center line of the tie-rod’s cross-section to avoid the effect of bending moment due to self-weight. If the anchor plates of the LVDTs are glued to the tie-rod, the tie-rod should be cleaned to remove any paint coating, and the glue should be only a thin line at the center line of the tie-rod’s cross-section.

To excite the tie-rod, a vertical impact by hand can be applied approximately every 5 seconds at different locations along the tie-rod. This is for the output-only modal analysis tests, in which the amount of the impact will not be measured. The duration of one test can be 60 seconds. At a same condition, three tests should be performed to calculate the average FRS. The applied vertical impact should not be too strong and should make minimum contact to the tie-rod in short time to avoid imposing damping effect on the tie-rod.

During the performance of the tests, the deflections at several points along the tie-rod should be measured to assess the effect of bending curvature due to self-weight and then compare with the numerical results.

7.2.2 Recommendation for the development of techniques to identify in-situ the tensile force in tie-rods

- More conservative technique

The technique is based on a frequency-based identification method that minimizes the measurement error. In particular, the first five to seven natural frequencies of the tie-rods are experimentally identified by measuring the FRS with manually applied vertical impact. Then, a numerical model is developed for axially loaded tie-rod in FE program (DIANA, 2008). The tie-rod is assumed to be a beam with uniform cross-section and supported by rotational
springs at both ends. The tie-rod is straight ignoring the effect of bending curvature due to self-weight. The tensile stress in the tie-rod, rotational stiffness of the supports and elastic modulus are the unknowns. The numerical model is used to calculate the frequencies for a given set of unknowns. After that, the identified frequencies are calibrated with the calculated frequencies by using the optimization process in which the frequency of the first mode is not considered. The optimization process minimizes the errors between the identified and calculated frequencies and gives the corresponding values of the unknowns including the tensile stress. The error in the values of the tensile stress is also given.

This technique is similar to the technique presented by Amabili et al. (2010). The difference is it should take into account the non-uniform section of ancient tie-rods and the portions of tie-rods clamped inside the masonry walls. Therefore, the technique has area for further improvement. However, a development in this technique is the exclusion of the frequency of the first mode, in order to obtain more accurate results for all types of tie-rods and in all situations. If the frequency of the first mode is included, in some cases, the effect of bending curvature due to self-weight is significant on the first mode and will affect the results. This technique is conservative because it provides higher tensile stress. Its advantage is of simple execution.

- More accurate technique

The technique is similar to the above described technique. However, it is more accurate and complete in acquiring the true dynamic behavior of tie-rods taking into account the effect of bending curvature due to self-weight. The numerical model is not assumed as a straight tie-rod but a deflected one due to self-weight or other additional weights and the tensile stress. During the calibration process between the identified and calculated frequencies, the model is run by geometric non-linear analysis to obtain the deflected shape. Then the geometry of the model is updated with that of the deflected shape. Then, the modal analysis is performed to determine the natural frequencies. The optimization process gives the values of the three unknowns, the deflections and the estimation errors of the unknowns. It should be noted that when carrying out the tests, the deflections of tie-rods should be measured to verify the numerical results. Overall, this technique is expected to provide the most accurate results.

Further work should be performed in the future include: (1) to improve the technique including bending curvature due to self-weight using geometric non-linear analysis to obtain the most accurate deflected shapes of tie-rods; and (2) to perform experimental tests for tie-rods in existing structures.
References


Annex A

A.1 Overview of Experimental Setup

Figure A.1 – Tests of tie-rod S2 with boundary condition (BC) type PP without the wireless accelerometer system and LVDTs at the right end of tie-rod
Figure A.2 – Tests of tie-rod S2 with boundary condition (BC) type FF

Figure A.3 – Tests of tie-rod C4 with boundary condition (BC) type FF
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Figure A.4 – Tests of tie-rod C3 with boundary condition (BC) type PP

Figure A.5 – Tests of tie-rod C3 with boundary condition (BC) type FF
A.2 Experimental Results

Table A-1
Comparison of the frequency values of tests S2-PP with numerical results excluding the effect of bending curvature due to self-weight

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\sigma_0$ (MPa)</th>
<th>$\sigma = \sigma_0 + \Delta \sigma$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.64</td>
<td>4.42</td>
</tr>
<tr>
<td>3</td>
<td>28.62</td>
<td>30.32</td>
</tr>
<tr>
<td>4</td>
<td>49.74</td>
<td>52.76</td>
</tr>
<tr>
<td>5</td>
<td>76.38</td>
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<td>116.02</td>
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<tr>
<td>7</td>
<td>146.35</td>
<td>155.80</td>
</tr>
</tbody>
</table>

Table A-2
Comparison of the frequency values of tests S2-FF with numerical results excluding the effect of bending curvature due to self-weight

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\sigma_0$ (MPa)</th>
<th>$\sigma = \sigma_0 + \Delta \sigma$ (MPa)</th>
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<td>6</td>
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<tr>
<td>7</td>
<td>152.45</td>
<td>155.13</td>
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</table>
Identification of the Tensile Force in Tie-rods using Modal Analysis Tests

Table A-3
Comparison of the frequency values of tests C3-PP with numerical results excluding the effect of bending curvature due to self-weight

<table>
<thead>
<tr>
<th>M</th>
<th>f (Hz)</th>
<th>(\sigma_0) (MPa)</th>
<th>(\sigma = \sigma_0 + \Delta\sigma) (MPa)</th>
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<tbody>
<tr>
<td>e</td>
<td>PP</td>
<td>PP</td>
<td>(%)</td>
</tr>
<tr>
<td>1</td>
<td>11.3</td>
<td>8.1</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>28.0</td>
<td>26.4</td>
<td>-6</td>
</tr>
<tr>
<td>3</td>
<td>54.6</td>
<td>56.5</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>83.7</td>
<td>108.3</td>
<td>29</td>
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<td>5</td>
<td>131.7</td>
<td>152.4</td>
<td>16</td>
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<td>6</td>
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Table A-4
Comparison of the frequency values of tests C3-FF with numerical results excluding the effect of bending curvature due to self-weight

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<th>f (Hz)</th>
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<th>(\sigma = \sigma_0 + \Delta\sigma) (MPa)</th>
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<td>PP</td>
<td>PP</td>
<td>(%)</td>
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<td>-13</td>
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Annex A

Table A-5
Comparison of the frequency values of tests C4-PP with numerical results excluding the effect of bending curvature due to self-weight

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<tbody>
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<td>f (Hz)</td>
<td>σ₀ (MPa)</td>
<td>σ = σ₀ + Δσ (MPa)</td>
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<td></td>
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<td>7.7 MPa</td>
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<td>46</td>
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Table A-6
Comparison of the frequency values of tests C4-FF with numerical results excluding the effect of bending curvature due to self-weight

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A.3 Zoom of the Proposed Standard Chart II-b at Low Tensile Stresses

Figure A.6 – Zoom of the proposed standard Chart II-b at low tensile stresses